GEOMETRIC TOMOGRAPHY, SECOND EDITION CORRECTIONS AND UPDATE

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1. Preface

Many years have passed since the publication of the second edition of *Geometric Tomography* and much has happened. One major event was the appearance in 2014 of the second edition of Schneider's classic text [227], which for some time lessened the pressure to post an update to my own book. But such has been the pace of advances in convex geometry and related areas that in 2020, when the first version of this update was posted, it finally became necessary to take action.

To keep the task manageable, the update is limited, at least for now, to a list of corrections and a set of reports on the problems stated at the end of the chapters. The idea is to post new versions of this manuscript, each numbered and dated, and widen the scope as time and energy allow, for example by adding reports on other open problems in geometric tomography.

A brief overview of progress on the problems can be found at the beginning of Section 4. There are many, particularly from Chapters 1, 2, 5, and 6, for which there is little or nothing to say. In fact, the problems vary widely in perceived importance. At one extreme stand the slicing problem, Problem 8.3—solved in 2024, after a truly remarkable series of advances—and Mahler's conjecture, Problem 9.2, whose significance and notoriety, not just in convex geometry but in mathematics as a whole, continue to increase steadily. At the other, there are problems that arose in the course of writing the first edition of the book and seemed potentially accessible to undergraduate and Masters students. These were motivated by the study of X-rays of convex and star bodies, a topic that although not the focus nowadays of much attention, does lend itself to natural problems, some of which may well await beautiful solutions. For example, Problem 2.1 is still untouched, despite appearing as Question 1 in my 1995 *AMS Notices* article on geometric tomography.

The two problems singled out above, the slicing problem and Mahler's conjecture, present by far the greatest challenge in adequately summarizing their status. This is due to the invasion of methods, results, and conjectures from other areas of geometry, such as contact geometry and symplectic geometry, and from reaches of analysis beyond those encountered in the book. Hundreds of pages have already been devoted to them in expository works, several of which are masterly and go into far more depth than the short reports presented here. Nevertheless, there does not appear to be any single source in the literature that provides a comprehensive overview of either problem. Figure 1 on page 13 raises questions which, while

obvious, have not appeared in the literature. For example, is KLS equivalent to its centered (i.e., origin-symmetric) version, thereby eliminating the need for the two dotted arrows? Are there any other relations between the 40 conjectures in the diagram? In particular, is it true that SISO2 $\Rightarrow$ MAH2, a relation that would connect the graph?

Rolf Schneider's eagle eye spotted a number of typos and this help is much appreciated. I am also very grateful to the following for helpful correspondence related to the specified aspects of this update: Juan Carlos Álvarez Paiva (Problem 9.2), Florent Balacheff (Problem 9.2), Paolo Dulio (Problems 1.5 and 5.6), Apostolos Giannopoulos (Problem 8.3), David Jerison (Problem 8.3), Roman Karasev (Problem 9.2), Bo'az Klartag (Problems 8.3 and 9.2), Alex Koldobsky (Problems 7.6 and 8.3), Emanuel Milman (Problems 8.3 and 9.3 to 9.5), Luis Montejano (Problems 3.3 and 7.4), Boris Rubin (Problem 8.5), Dmitry Ryabogin (Problems 3.1, 3.2, 7.2, and 7.3), Grzegorz Sójka (correction for p. 52), Yanir Rubinstein (Problems 8.3 and 9.2), Gaoyong Zhang (Problems 4.4, 8.3, 8.6, 8.7, 9.2, and 9.3), and Artem Zvavitch (Problem 9.2).

Any further mistakes or misprints, as well as news about the open problems, can be sent to `Richard.Gardner@wwu.edu`. A current version of this document will be kept at `http://faculty.wwu.edu/gardner/` (click on Research).

2. Remarks on terminology, notation, and names

In preparing the second edition, I felt some obligation to purchasers and readers of the first edition not to make too many changes in basic terminology and notation. However, even then my taste had moved with the times, and since its appearance in 2014, the second edition of Rolf Schneider's classic text also has to be taken into account. Regarding notation, nowadays I prefer:

1. $\mathbb{R}^n$ rather than $\mathbb{E}^n$ for n-dimensional Euclidean space.
2. B^n exclusively (and not B) for the unit ball in $\mathbb{R}^n$.
3. ∂E for the boundary of E instead of $\operatorname{bd} E$.
4. $\mathcal{K}^n_n$ instead of $\mathcal{K}^n_0$ for the class of convex bodies in $\mathbb{R}^n$.
5. $\mathcal{H}^n$ rather than λ_n for n-dimensional Lebesgue or Hausdorff measure, and V or V_n when the set in question is a body.
6. K° instead of K^* for the polar body of K, though the latter is consistent with the notation for the dual space in functional analysis.

In terminology, I now favor:

1. *o-symmetric* or *origin symmetric* rather than *centered* (see p. 3 of the book). Unfortunately, "centered" is sometimes used to mean having centroid at the origin. (The term *centrally symmetric* seems increasingly to be abused; its meaning is clear and is not the same as origin symmetric.)
2. *dilation* instead of *dilatation* (see p. 5 of the book), as almost everyone else does, even though technically the latter is correct.

Despite these current preferences, notation and terminology in what follows will generally adhere to that in the book, and the meaning of symbols and terms can be found there.

A word about names. Several family names, such as Milman and Zhang, are shared by more than one individual in the community. Nevertheless, only the last name is given when citing an author in the text, since the initials are available in the bibliography.

3. Corrections and amendments

Page	**Line**	**Comments**
52	1–2	It has come to my attention that "all points in V are isolated" does not contradict Lemma 1.2.26, as claimed. Thus the proof of Lemma 1.2.27, if the statement is true, is unfortunately incomplete.
54	-2	Replace "*(a)*" by "*(b)*".
110	14	The proof of Theorem 3.2.7 should start as follows: We may assume without loss of generality that the center of K is at the origin. Then origin symmetry implies that $h_K(u) = h_K(-u)$, and...
126	-12	Replace "constants a and b" by "$a, b > 0$".
128	-15	Insert "'and $k = n - 1$" after "odd".
133	13	The small indent at the left should be removed.
133	-2	Replace "$i = 1$, $j = 2$" by "$i = 1$, $j = k = 2$, $n = 3$".
134	2–9	The results of Howard and Hug were actually obtained under the (formally) stronger assumption that the ith and jth projection functions of K_1 and K_2 are proportional, in other words, $V_i(K_1|S) = a\, V_i(K_2|S)$, for all $S \in \mathcal{G}(n,i)$, and $V_j(K_1|T) = b\, V_j(K_2|T)$, for all $T \in \mathcal{G}(n,j)$.
185	-12	The fact that the unit cube in $\mathbb{E}^n$ has property (VP) was proved independently by Chakerian and Filliman [56] and attributed by them to Michael Kallay.
204	6–7	Delete "*, if* $i \geq 0$*, and* $\nu_k(E_1) > \nu_k(E_2)$*, if* $i < 0$*"*.
205	13–15	Replace this sentence by "If $i < 0$, the same conclusion is reached after noting that in this case $s_j(\theta)^i \leq r_j(\theta)^i$, for $j = 1, 2$, and there is an extra change of sign in the integrand."
213	Fig. 5.9	The bold and dotted lines bounding the component C should be interchanged. As it is, both C and p_3C are in $K \setminus K'$.
226	20	Insert before the last sentence in the first paragraph of Note 5.1: "When $p = o$ and E is a bounded domain containing a ball with center o, the directed chordal symmetral D_pE of a set E at a point p was first defined by Bandle and Marcus [27], who called it the *radial concentration* of E and utilized it in the theory of capacities. Their definition was actually more general, allowing the use of some non-Euclidean metrics."
341	22–23	Replace "Problems 8.6 and 8.7" by "Problem 8.6" and replace "the hypotheses then imply" by "when $n \geq 3$, the stronger hypothesis $IL = cL$ for some $c > 0$ implies".

Page	Line	Comments
345	19–24	The results quoted on these lines only pertain to the special case of Theorem 8.2.13 when L is also a centered convex body. (For an update on the slicing problem, see below.)
373	-7	It might be noted that the cases $i = n-1$ and $i = 1$ of Corollary 9.4.8 are consequences of the equality conditions in Theorem 9.3.1 (right-hand inequality) and Theorem 9.3.2, respectively.
381	-8	Replace "case $p = 1$ was" by "cases $p = 1$ and $p = 2$ were".
385	-12	Insert ", the centroid of M is at the origin," after "$\lambda_n(M) = 1$".

UPDATED REFERENCES

[17] G. Averkov and G. Bianchi. Retrieving convex bodies from restricted covariogram functions. *Adv. in Appl. Probab.* **39** (2007), 613–29.
[18] M. Baake, P. Gritzmann, C. Huck, B. Langfeld, and K. Lord. Discrete tomography of planar model sets. *Acta Crystallogr. Sect. A* **62** (2006), 419–33.
[46] K. J. Batenburg. A network flow algorithm for reconstructing binary images from discrete X-rays. *J. Math. Imaging Vision* **27** (2007), 175–91.
[59] G. Bianchi. The covariogram determines three-dimensional convex polytopes. *Adv. Math.* **220** (2009), 1771–808.
[98] U. Brehm. X-ray problems with point sources for grid polygons. Unpublished manuscript.
[133] D. Butcher, A. Medin, and D. C. Solmon. Planar convex bodies with a common directed X-ray. *Rend. Istit. Mat. Univ. Trieste* **37** (2005), 181–206 (2006).
[149] S. Campi and P. Gronchi. On volume product inequalities for convex sets. *Proc. Amer. Math. Soc.* **134** (2006), 2393–402.
[192] P. Dulio. Geometric tomography in a graph. *Rend. Circ. Mat. Palermo* (2) **77** (2006), 229–66.
[197] P. Dulio and C. Peri. Point X-rays of convex bodies in planes of constant curvature. *Rocky Mountain J. Math.* **36** (2006), 915–30.
[198] P. Dulio and C. Peri. On the geometric structure of lattice U-polygons. *Discrete Math.* **307** (2007), 2330–40.
[273] R. J. Gardner and M. Kiderlen. A solution to Hammer's X-ray reconstruction problem. *Adv. Math.* **214** (2007), 323–43.
[274] R. J. Gardner, M. Kiderlen, and P. Milanfar. Convergence of algorithms for reconstructing convex bodies and directional measures. *Ann. Statist.* **34** (2006), 1331–74.
[324] P. Goodey and W. Weil. Directed projection functions of convex bodies. *Monatsh. Math.* **149** (2006), 43–64.
[325] P. Goodey and W. Weil. Determination of convex bodies by directed projection functions. *Mathematika* **53** (2006), 49–69 (2007).
[361] P. Gronchi and M. Longinetti. Affinely regular polygons as extremals of area functionals. *Discrete Comput. Geom.* **39** (2008), 273–97.
[400] R. Howard. Convex bodies of constant width and constant brightness. *Adv. Math.* **204** (2006), 241–61.
[401] R. Howard and D. Hug. Smooth convex bodies with proportional projection functions. *Israel J. Math.* **159** (2007), 317–41.
[402] R. Howard and D. Hug. Nakajima's problem: Convex bodies of constant width and constant brightness. *Mathematika* **54** (2007), 15–24.
[431] M. Kiderlen. Blaschke and Minkowski endomorphisms of convex bodies. *Trans. Amer. Math. Soc.* **358** (2006), 5539–64.

[442] D. A. Klain. On the Minkowski condition and divergence. Unpublished manuscript available at `http://faculty.uml.edu/dklain/mpdiv.pdf`.
[469] A. Koldobsky and C. Shane. The determination of convex bodies from derivatives of section functions. *Arch. Math. (Basel)* **88** (2007), 279–88.
[470] A. Koldobsky, V. Yaskin, and M. Yaskina. Modified Busemann–Petty problem on sections of convex bodies. *Israel J. Math.* **154** (2006), 191–207.
[527] M. Ludwig. Intersection bodies and valuations. *Amer. J. Math.* **128** (2006), 1409–28.
[617] E. Milman. Dual mixed volumes and the slicing problem. *Adv. Math.* **207** (2006), 566–98.
[618] E. Milman. A comment on the low-dimensional Busemann–Petty problem. In *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 1910. Springer, Berlin, 2007, pp. 245–53.
[619] E. Milman. Generalized intersection bodies. *J. Funct. Anal.* **240** (2006), 530–67.
[635] E. Morales-Amaya and L. Montejano. Variations of classic characterizations of ellipsoids and a short proof of the false centre theorem. *Mathematika* **54** (2007), 35–40.
[706] B. Rubin. The lower dimensional Busemann–Petty problem with weights. *Mathematika* **53** (2006), 235–45 (2007).
[743] R. Schneider. Stable determination of convex bodies from projections. *Monatsh. Math.* **150** (2007), 241–7.
[785] A. Stancu. The floating body problem. *Bull. London Math. Soc.* **38** (2006), 839–46.
[848] V. Yaskin. The Busemann–Petty problem in hyperbolic and spherical spaces. *Adv. Math.* **203** (2006), 537–53.
[849] V. Yaskin. A solution to the lower-dimensional Busemann–Petty problem in the hyperbolic space. *J. Geom. Anal.* **16** (2006), 735–45.
[850] V. Yaskin and M. Yaskina. Centroid bodies and comparisons of volumes. *Indiana Univ. Math. J.* **55** (2006), 1175–94.
[851] M. Yaskina. Non-intersection bodies all of whose central sections are intersection bodies. *Proc. Amer. Math. Soc.* **135** (2007), 851–60.

4. Reports on the problems

Citations such as [153book] refer to the references in the book, and numbers associated to pages, equations, theorems, or notes without reference to another work refer to those in the book. A brief overview is as follows, where for problems not listed, any progress is detailed in the full reports below.

Solved: Problems 8.3, 8.8, 8.9(i), 9.3.

Solved but with modified versions open: Problems 3.1, 3.2, 7.2, 7.3.

Solved in significant special cases: Problem 3.3 (for arbitrary n and $k \equiv 1 \pmod 4$, $k \neq 133$), Problem 3.9 (assuming that K_1 and K_2 have proportional ith and jth projection functions, $1 \leq i < j \leq n-2$, $(i,j) \neq (1, n-2)$, and K_2 enjoys a certain weak regularity condition), Problem 7.4 (when L is a convex body containing o, n is arbitrary, and $i \equiv 1 \pmod 4$, $i \neq 133$), Problem 7.6 (when L_1 and L_2 are polytopes), Problems 8.6 and 8.7, when $i = n-1$ and "homothetic to" is replaced by "a dilatate of," Problem 9.2 (in its centered version, when $n = 3$), Problem 9.4 (when $i = 1$).

Problem 1.1

The problem is also stated in [70]. One may as well take the two X-rays in the coordinate directions. Vincze and Nagy [233] define a map Φ that takes a planar compact convex set K to its *generalized conic function* $f_K(x)$, defined as the integrated taxicab distance of x to the points in K. Up to a constant, the second partial derivatives of f_K are the X-rays of K in the coordinate directions. In [233, Theorem 5], it is shown that K is determined by its X-rays in the coordinate directions if and only if the set-valued inverse of Φ is lower semi-continuous at f_K, and in [233, Remark 3] the authors speculate on how this condition might be utilized.

Problem 1.5

The problem remains unsolved, but there are results dealing with the case when $\varepsilon = 0$. Dulio, Longinetti, Peri, and Venturi [71, Theorem 1.1] prove that if $K, L \in \mathcal{K}_0^2$ and $X_{u_i}K = X_{u_i}L$ for $i = 1, \dots, m$, $m \geq 3$, then

$$\lambda_2(K \triangle L) \leq \frac{1 - \cos(\pi/m)}{\cos(\pi/m)} \lambda_2(K \cap L),$$

with equality if and only if, up to a nonsingular affine transformation, K and L are regular m-gons, K is a rotation of L about o by π/m, and $\{u_1, \dots, u_m\}$ are the corresponding equally spaced directions. Some refinements and variations are also given in [71], including estimates in terms of cross ratios, when $m \geq 4$, as well as when the directions are only known up to a fixed error.

Problem 2.2

The problem is also stated in [70].

Problem 2.8

The problem in the form stated is still open. However, it is not true that if $\mathcal{S}$ is a finite set of directions in $\mathbb{E}^2$ such that planar convex bodies are determined by their X-rays in the directions in $\mathcal{S}$, then planar convex bodies are also determined among measurable sets by their X-rays in the directions in $\mathcal{S}$. Indeed, let $\mathcal{S} = \{(1,0), (0,1), (2,1), (-1,2)\}$. It was proved by Gardner and Gritzmann [267book, Theorem 6.2(i)] that planar convex bodies are determined by their X-rays in the directions in $\mathcal{S}$. We claim that if $K = \operatorname{conv}\{(2,4), (-2,-4), (4,-2), (-4,2)\}$, a parallelogram, then there are an $r > 1$ and a measurable set E in $\mathbb{E}^2$ essentially different from rK such that rK and E have the same X-rays in the directions in $\mathcal{S}$. To see this, we use an $\mathcal{S}$-switching component discovered by Gritzmann, Langfeld, and Wiegelmann [98]. In [98, Fig. 1, p. 1594], two finite lattice sets are depicted. Denoting the left set by A and the right set by B, we see that A consists of the integer lattice points contained in the parallelogram K. If $F = A \setminus B$ and $G = B \setminus A$, then $|F| = |G| = 8$ and $F \cup G$ is an $\mathcal{S}$-switching component. Moreover, $F \subset \operatorname{bd} K$ and $G \subset \mathbb{E}^2 \setminus K$. Since G is finite, we may choose $r > 1$ such that $G \subset \mathbb{E}^2 \setminus rK$ and of course $F \subset \operatorname{int} rK$. Now let $\varepsilon > 0$ be small enough that $F + \varepsilon B^2 \subset \operatorname{int} rK$ and $G + \varepsilon B^2 \subset \mathbb{E}^2 \setminus rK$. Setting $E = (rK \setminus (F + \varepsilon B^2)) \cup (G + \varepsilon B^2)$, it is clear that E is a measurable set with the same X-rays as rK in the directions in $\mathcal{S}$.

Problem 3.1

Note the restriction $2 < k < n-1$, which eliminates $\mathbb{E}^3$ as a setting for this problem. The reason is that the Petty–McKinney example in Theorem 3.1.8 already gives a negative answer when the projections are 2-dimensional. In any case, the examples constructed by Zhang [238, Theorem 1.4] answer this problem negatively, even when the dilatation factor for each projection is one. See the report for Problem 3.2 for more details.

It seems that the following modified version of Problem 3.1, inspired by [238, Remark 4.2], is open.

Suppose that $2 < k \leq n-1$, and that K_1 and K_2 are convex bodies in $\mathbb{E}^n$ such that $K_1|S$ is similar to $K_2|S$ for all $S \in \mathcal{G}(n,k)$. Is K_1 equal to K_2 up to a dilatation, a translation, and a reflection in a proper subspace?

Problem 3.2

Zhang [238, Theorem 1.4] solves this problem negatively. However, the bodies he constructs are reflections of each other in a nontrivial proper subspace.

The following positive partial results have been obtained.

Myroshnychenko and Ryabogin [190] show that the answer is affirmative if one of the bodies is a polytope.

Ryabogin [217] proves that if $f, g \in C(S^{n-1})$, $n \geq 3$, and for each $S \in \mathcal{G}(n,2)$, there is a rotation ϕ_S in S about o such that $f(\phi_S u) = g(u)$ for each $u \in S^{n-1} \cap S$, then $f(u) = g(\pm u)$ for each $u \in S^{n-1}$. As a corollary, he concludes that if $K|S$ is a rotation in S about o of $L|S$ for each $S \in \mathcal{G}(n,2)$, then $K = \pm L$. The case $n = 3$ of the latter result was also obtained by Mackey [170].

Extra conditions can be placed on K and L that ensure that the desired conclusion, or a slightly modified version of it, is true. These conditions stipulate the existence of diameters or forbid symmetries in the projections. The details are rather involved, so we refer the reader to Ryabogin [218], Alfonseca, Cordier, and Ryabogin [7], and the references given in these papers.

The following modified versions of Problem 3.2 remain open. The first was posed by Ryabogin (private communication) and the second is taken from [238, Remark 4.2].

Suppose that K_1 and K_2 are convex bodies in $\mathbb{E}^3$ such that $K_1|S$ is directly congruent to $K_2|S$, for all $S \in \mathcal{G}(3,2)$. Is K_1 a translate of $\pm K_2$?

Here, *directly congruent* means equal up to a direct rigid motion.

Suppose that $2 < k \leq n-1$, and that K_1 and K_2 are convex bodies in $\mathbb{E}^n$ such that $K_1|S$ is congruent to $K_2|S$ for all $S \in \mathcal{G}(n,k)$. Are K_1 and K_2 equal, up to a translation and a reflection in proper subspace?

Ryabogin [219] surveys Problem 3.2 and poses other questions of a similar nature; answers to some of these have been found by Myroshnychenko, Ryabogin, and Saroglou [191] and Zhang [238, p. 2065].

Problem 3.3

Montejano [188] supplies an affirmative answer when n is arbitrary and $k \equiv 1 \,(\mathrm{mod}\ 4)$, $k \neq 133$. This leaves open the cases when n is arbitrary and $k = 133$ or $k \equiv 3 \,(\mathrm{mod}\ 4)$. In his proof, Montejano uses some of the results in [46], which solved the corresponding cases of Problem 7.4, the question for sections dual to Problem 3.3.

Montejano [189] provides a detailed commentary on [188] and also obtains an affirmative answer when $k = n - 2$ is odd, under the stronger assumption that all the projections of K are linearly equivalent; see [189, Theorem 5.8].

See the report for Problem 7.4 for further comments.

Problem 3.9

Howard and Hug [401book], [402book] obtained partial solutions to Problem 3.9 under the (formally) stronger assumption that

$$V_i(K_1|S) = a\, V_i(K_2|S), \quad \forall S \in \mathcal{G}(n,i)$$

and

$$V_j(K_1|T) = b\, V_j(K_2|T), \quad \forall T \in \mathcal{G}(n,j),$$

in other words, the ith and jth projection functions of K_1 and K_2 are proportional. (It is stronger by Kubota's integral recursion (A.46), p. 408.) See p. 134 and the corrections listed above in this update. For convenience, if not historical accuracy, we shall refer to the special case when $K_2 = B^n$ of this modified version of Problem 3.9 as *Nakajima's problem*. This includes the question as to whether a convex body of constant width and constant brightness must be a ball. Building on earlier work, Hug [113] obtains an affirmative answer to the same modified version of Problem 3.9, when $1 \leq i < j \leq n-2$, $(i,j) \neq (1, n-2)$, and K_2 enjoys a certain weak regularity condition (satisfied, for example, when $\mathrm{bd}\, K_2$ contains a small region in which it is C^2_+). It follows that Nakajima's problem has an affirmative answer for these values of i and j, but remains open for arbitrary K_1 when $(i,j) = (1, n-2)$ and $n \geq 6$, and when $1 \leq i \leq n-2$, $j = n-1$, and $n \geq 4$. More is known when $K_1 \in C^2$ but there has been no progress beyond what is already stated in Note 3.6.

Problem 4.4

Saroglou and Zvavitch [226] show that if $n \geq 3$ and there is a $\phi \in GL(n)$ such that the curvature function $f_{\phi K}$ exists and is sufficiently close to 1 (i.e, $\|f_{\phi K} - 1\|_\infty \leq \varepsilon$ for some $\varepsilon > 0$), then $\Pi^m K \to B^n$ as $m \to \infty$ in the Banach–Mazur metric. From this they are able to conclude that if K satisfies these conditions and $\Pi^2 K$ is homothetic to K, then K is an ellipsoid. The latter result was obtained independently by Ivaki [119] with an extra smoothness assumption on K.

It is already mentioned in Note 4.6 that when $i = 1$, Schneider [732book] proves that K must be a ball. Ivaki [118] addresses Problem 4.4 for the case when $1 < i < n-1$ and shows that with the same assumptions on K as in [119], if $\Pi_i^2 K$ is homothetic to K, then K is a ball.

Ivaki's results mentioned above are generalized to sufficiently regular even Minkowski valuations by Ortega-Moreno and Schuster [198].

Though they do not bear directly on Problem 4.4, related results of Saraglou [225] are worth mentioning. He proves that if $K \in \mathcal{K}_0^3$, then $\Pi^2 K \subset 8\lambda_3(K)K$ if K is a zonoid and $6\lambda_3(K)K \subset \Pi^2 K$ if K is centrally symmetric, both inclusions being sharp.

Problem 5.6

The problem as stated is completely open, but Dulio, Longinetti, Peri, and Venturi [71, Section 6] apply their results on Problem 1.5, together with Theorem 6.2.8, to obtain stability estimates for planar convex bodies with equal -1-chord functions at finitely many points.

Problem 7.2

This is the dual version of Problem 3.1. The examples constructed by Zhang [238, Theorem 1.4] provide a negative answer. See the reports for Problems 3.1, 3.2, and 7.3 for more details.

Problem 7.3

This is the dual version of Problem 3.2. All the results, both positive and negative, reported for Problem 3.2 have corresponding versions for Problem 7.3 and are stated in the articles referred to for Problem 3.2. Suitably modified versions of Problem 7.3 may also be posed, in analogy to those for Problem 3.2; of course, translation is not permitted in this case.

Problem 7.4

This is the dual version of Problem 3.3. For convex L containing o, the problem is attributed to Banach [26] (see p. 244 of the French original or p. 152 of the English translation), where, of course, it is phrased in the language of Banach spaces.

Bor, Hernández Lamoneda, Jiménez-Desantiago, and Montejano [46] achieve a remarkable breakthrough by proving the following result.

Problem 7.4 has an affirmative answer when L is a convex body containing o, n is arbitrary, and $i \equiv 1 \,(\mathrm{mod}\ 4)$, $i \neq 133$.

In this setting (when L is convex containing o), this leaves open the cases when $i = n - 1$ and $i = 133$ or $i \equiv 3 \,(\mathrm{mod}\ 4)$. Apparently the reason for the exclusion of $i = 133$ stems from the fact that 133 is the dimension of the exceptional Lie group E_7; see [46, Theorem 1.6]. The authors of [46] use the following new result in their proof.

If $K \in \mathcal{K}_0^n$, $n \geq 3$, is centered and such that all its sections by hyperplanes are linearly equivalent affine images of bodies of revolution in their respective hyperplanes, then K is an ellipsoid.

In a long survey article with some new results and several proofs, Montejano [189] gives a detailed commentary on [46] and [188], and revisits and reworks Gromov's result and ideas in [359book] (see Note 7.2).

As far as I know, this problem is still completely open when L is a non-convex star body in $\mathbb{E}^n$.

Problem 7.6

Howard, Nazarov, Ryabogin, and Zvavitch [111] answer the question affirmatively for star bodies of revolution in $\mathbb{E}^3$ with C^1 radial functions. For the more general problem in $\mathbb{E}^n$, Yaskin [237] obtains an affirmative answer when L_1 and L_2 are convex polytopes.

Ryabogin and Yaskin [220, Theorem 2.4] show that the assumption that L_1 and L_2 are centered is necessary. Their result is as follows.

There are noncongruent convex bodies L_1 and L_2 in $\mathbb{E}^n$, containing the origin in their interiors, such that for $1 \le i \le k \le n-1$, we have

$$V_i(L_1 \cap S) = V_i(L_2 \cap S),$$

for all $S \in \mathcal{G}(n,k)$. Moreover, L_1 and L_2 can be constructed in such a way that both are C^∞_+ bodies of revolution or both are polytopes.

Note that the same result with V_i replaced by $\widetilde{V}_i$ is implied by Theorems 7.2.13 and 7.2.14.

The problem as stated, in all its versions, is still open.

Related to Problem 7.6 is the following Busemann-Petty problem for surface areas.

Let K_1 and K_2 be centered convex bodies in $\mathbb{E}^n$ such that

$$\lambda_{n-2}((\operatorname{bd} K_1) \cap u^\perp) \le \lambda_{n-2}((\operatorname{bd} K_2) \cap u^\perp),$$

for all $u \in S^{n-1}$. Is it true that $\lambda_{n-1}(\operatorname{bd} K_1) \le \lambda_{n-1}(\operatorname{bd} K_2)$?

The question was posed by König and Koldobsky [160], who showed that the answer is negative if K_1 is a unit cube and K_2 is a ball of suitable radius, provided $n \ge 14$. Brazitikos and Liakopoulos [51] consider the isomorphic version of this question, i.e., whether the hypotheses imply that $\lambda_{n-1}(\operatorname{bd} K_1) \le C\lambda_{n-1}(\operatorname{bd} K_2)$ for a universal constant C, along with versions of the slicing problem for surface area.

Problem 8.2

Rubin [215] shows that the answer is affirmative when the body with smaller sections is invariant under rotations preserving a pair of mutually orthogonal subspaces of dimensions k and $n-k$, $1 \le k \le n-1$ (the case $k=1$ corresponding to a body of revolution), satisfying $i \le \min\{k, n-k\}$. The problem remains open.

Problem 8.3

As was mentioned in Notes 8.9 and 9.8, Problem 8.3 is equivalent to the the slicing problem, posed by Bourgain in 1986. In 2024, following an advance due to Guan [100], Klartag and Lehec [143] announced a solution. This was the culmination of decades of amazing contributions and advances in understanding its relation to other problems. Schneider [227, pp. 605–6] presents a very brief summary of developments prior to 2014. In this report, we attempt to survey the history of the problem, its relation to other conjectures, and the story of its final solution. We shall refer both to the slicing problem and the *hyperplane conjecture* that the answer to the problem is affirmative, as indeed it turned out to be. The term *hyperplane problem* is also in use, but has the disadvantage of also applying to Banach's question as to whether every infinite-dimensional Banach space is isomorphic to its hyperplanes. It will be

convenient to formulate several other related problems as conjectures, without implying that anyone strongly believes them to be true.

Milman and Pajor's paper [621book], already cited in Note 9.8, was the first comprehensive study of the topic. In addition to the lecture notes of Giannopoulos [92], there are several other recent expositions, ranging from the pithy and timely advertisement by Klartag and Werner [148] to long chapters or whole books, by Alonso-Gutiérrez and Bastero [9], Artstein-Avidan, Giannopoulos, and Milman [16, Chapter 10], Brazitikos, Giannopoulos, Valettas, and Vritsiou [50], and Klartag and Milman [147]. It transpires that the slicing problem is related to several other major open problems, and we draw on all these sources and others in an attempt to describe succinctly the current state of the art. Figure 1 (compare [165, Fig. 2]) shows the known relations between the various conjectures. We now know that all eight conjectures on the lowest level, and several other equivalent ones besides, are true, but all the other conjectures above this level are still open. In addition, at the top levels some basic questions are unresolved, or have answers not yet published and fully understood only by a small group of experts.

The slicing problem or hyperplane conjecture originated with the work of Bourgain [47], motivated by harmonic analysis; see the remark after [47, Lemma 2], as well as [16, pp. 360–1], [50, p. 141], and [147] for more details. In the context of centered convex bodies, some basic information, including the equivalence of the hyperplane conjecture to the *isotropic constant conjecture* (ISO) that $L_K \le C$ for any isotropic centered convex body in $\mathbb{E}^n$ and a universal constant C (see (9.11), p. 385), is already set out in Note 9.8. However, both conjectures have versions that are apparently more general but turn out to be equivalent. Paouris [200] proved that we could also state the hyperplane conjecture without the symmetry assumption, as follows.

Hyperplane conjecture (HYP). There is a universal constant $C > 0$ such that if $K \in \mathcal{K}_0^n$ and $\lambda_n(K) = 1$, there is a hyperplane H such that $\lambda_{n-1}(K \cap H) > C$.

(Here, and in what follows, possibly different universal constants are all denoted by C.) For ISO, we first need to extend some basic concepts. An arbitrary $K \in \mathcal{K}_0^n$ is *isotropic* if it satisfies the conditions on p. 385 and in addition has its centroid at the origin. For isotropic convex bodies, L_K is defined as for centered bodies, and one observes that any $K \in \mathcal{K}_0^n$ has an isotropic image ϕK under some $\phi \in GA_n$. Its *isotropy constant* L_K can then be defined as the isotropy constant of this affine image. Explicit formulas for L_K, such as those in [227, p. 605] or (9) below, are available. With this understanding, ISO is equivalent to the statement that $L_K \le C$ for any convex body in $\mathbb{E}^n$ and a universal constant C. This follows from Klartag's observation in [131, Remark 2, p. 392] (see also [16, Proposition 10.2.15], [50, Proposition 2.5.10], or [92, Proposition 8.9]) that there is a universal constant C such that for any isotropic convex body K in $\mathbb{E}^n$, there is a centered convex body K_0 such that $L_K \le C\, L_{K_0}$.

In fact, ISO may also equivalently be formulated for log-concave measures. An account of the intriguing interplay between convex bodies and log-concave measures can be found in [227, Section 9.5] (see also [16, Section 10.2], [50, Section 2.5], or [92, Sections 7 and 8]). The core result is that if f is a nonnegative log-concave function on $\mathbb{E}^n$ with $f(o) > 0$ and finite

positive integral, and $p > 0$, then

$$K_p(f) = \{x \in \mathbb{E}^n : p \int_0^\infty f(rx)\, r^{p-1}\, dr \geq f(0)\} \tag{1}$$

defines a convex body in $\mathbb{E}^n$. See, for example, [50, Proposition 2.5.5] or [284book, Corollary 4.2]; this is due to Ball, who stated it in [23, p. 74], [20book] for even f and $p \geq 1$. Moreover, K is centered if f is even, and if K is a convex body with $o \in \operatorname{int} K$, then $K_p(1_K) = K$. In fact, Klartag's observation [131, Remark 2, p. 392], [50, Proposition 2.5.10] mentioned in the previous paragraph uses this construction, since one can take $K_0 = K_{n+2}(f)$, where $f = g_K$ is the covariogram of K defined on p. 377, which is easily shown to be log-concave by the Brunn–Minkowski inequality, and satisfies $g_K(o) = \lambda_n(K) > 0$. See also [175], where Martín-Goñi finds for each n the optimal constant d_n such that $L_K \leq d_n L_{K_{n+2}(g_K)}$.

Here we take a *log-concave measure* to be a probability measure in $\mathbb{E}^n$ with a log-concave density function f. Such a measure μ is called *isotropic* if it has its centroid at the origin and

$$\int_{\mathbb{E}^n} (u \cdot x)^2\, d\mu(x) = 1 \tag{2}$$

for all $u \in S^{n-1}$, and its *isotropic constant* is defined by

$$L_\mu = \sup_{x \in \mathbb{E}^n} f(x)^{1/n}.$$

Then ISO, and therefore HYP, can be reinterpreted as asking whether there is a universal constant C such that $L_\mu \leq C$ for all such μ. This is because an extension of a result of Ball [20book] in [16, Proposition 10.2.16], [50, Proposition 2.5.12] or [92, Proposition 8.10] shows that if μ is an isotropic log-concave measure in $\mathbb{E}^n$ with density f, then there are universal constants $c_1, c_2 > 0$ such that

$$c_1 L_\mu \leq L_{K_{n+1}(f)} \leq c_2 L_\mu,$$

where $K_{n+1}(f)$ is the convex body defined by (1) with $p = n + 1$.

To summarize, in all their versions, we have

$$\mathsf{HYP} \Leftrightarrow \mathsf{ISO}.$$

(A proof of this equivalence is also given in [50, Theorem 3.1.2 and pp. 107–8], with the help of several results from [50, Section 2.2].)

Functional versions of ISO have been considered and are extensively studied by Fradelizi and Marín Sola [81]. If f is a nonnegative integrable log-concave function on $\mathbb{E}^n$ with nonzero integral, one can define its isotropic constant by

$$L_f = \left(\frac{\max_{x \in \mathbb{E}^n} f(x)}{\int_{\mathbb{E}^n} f(x)\, dx} \right)^{1/n} (\det \operatorname{Cov}(f))^{1/2n}, \tag{3}$$

where $\operatorname{Cov}(f)$ is the covariance matrix of f. (When f is the density function of an isotropic log-concave measure, L_f equals the isotropic constant L_μ of μ defined above.) For reasons

explained below after the introduction of the sharp isotropic constant conjectures, a preference is expressed in [81] for a different definition, namely

$$\widehat{L}_f = \left(\frac{e^{-h(f)}}{\int_{\mathbb{E}^n} f(x)\,dx} \right)^{1/n} (\det \mathrm{Cov}(f))^{1/2n}, \tag{4}$$

where h is the *differential entropy* of f, i.e.,

$$h(X) = -\left(\int_{\mathbb{E}^n} f(x) \log f(x)\,dx \right) / \int_{\mathbb{E}^n} f(x)\,dx. \tag{5}$$

Each definition of the isotropic constant of f gives rise to a functional version of ISO, namely that it is bounded above by a universal constant. It follows from [81, Remark 3.1] that each is equivalent to HYP.

Henceforth we shall give preference to HYP in stating its consequences, even when they follow more immediately from ISO.

The discussion in Note 9.8 mentions a couple of other statements equivalent or related to HYP, for example one due to Klartag and Milman [443book] involving Steiner symmetrization, to which should be added a result in a note of Ball [24]. Others still were already known to Milman and Pajor [621book]. In addition to some of these discussed below, there is a connection to the following stronger form of Milman's reverse Brunn–Minkowski inequality (see [16, Theorem 8.4.3] and [227, p. 380]).

Reverse Brunn–Minkowski inequality conjecture (RBM). There is a universal constant $C > 0$ such that if $K, L \in \mathcal{K}_0^n$ are isotropic, then

$$\lambda_n(K+L)^{1/n} \le C \left(\lambda_n(K)^{1/n} + \lambda_n(L)^{1/n} \right).$$

In [621book, pp. 78–9] (see also [50, p. 111]), it is shown that HYP $\Rightarrow$ RBM and credit is given to Ball for first proving this. (The reference in [621book] to Ball's 1986 PhD thesis does not appear to be correct.) Bourgain, Klartag, and Milman [48, Proposition 1.4] (see also [50, Theorem 3.2.4]) prove the reverse implication, so in fact

$$\text{HYP} \Leftrightarrow \text{RBM}.$$

Another statement from Milman and Pajor [621book] is phrased in the following slightly different form in [147].

Ellipsoid intersection conjecture (ELL). There is a universal constant C such that if $K \in \mathcal{K}_0^n$, there is an ellipsoid $E \subset \mathbb{E}^n$ with $\lambda_n(E) = \lambda_n(K)$ such that

$$\lambda_n(K \cap C\,E) \ge \lambda_n(K)/2.$$

In [621book, Proposition 5.5], it is proved that

$$\text{HYP} \Leftrightarrow \text{ELL}.$$

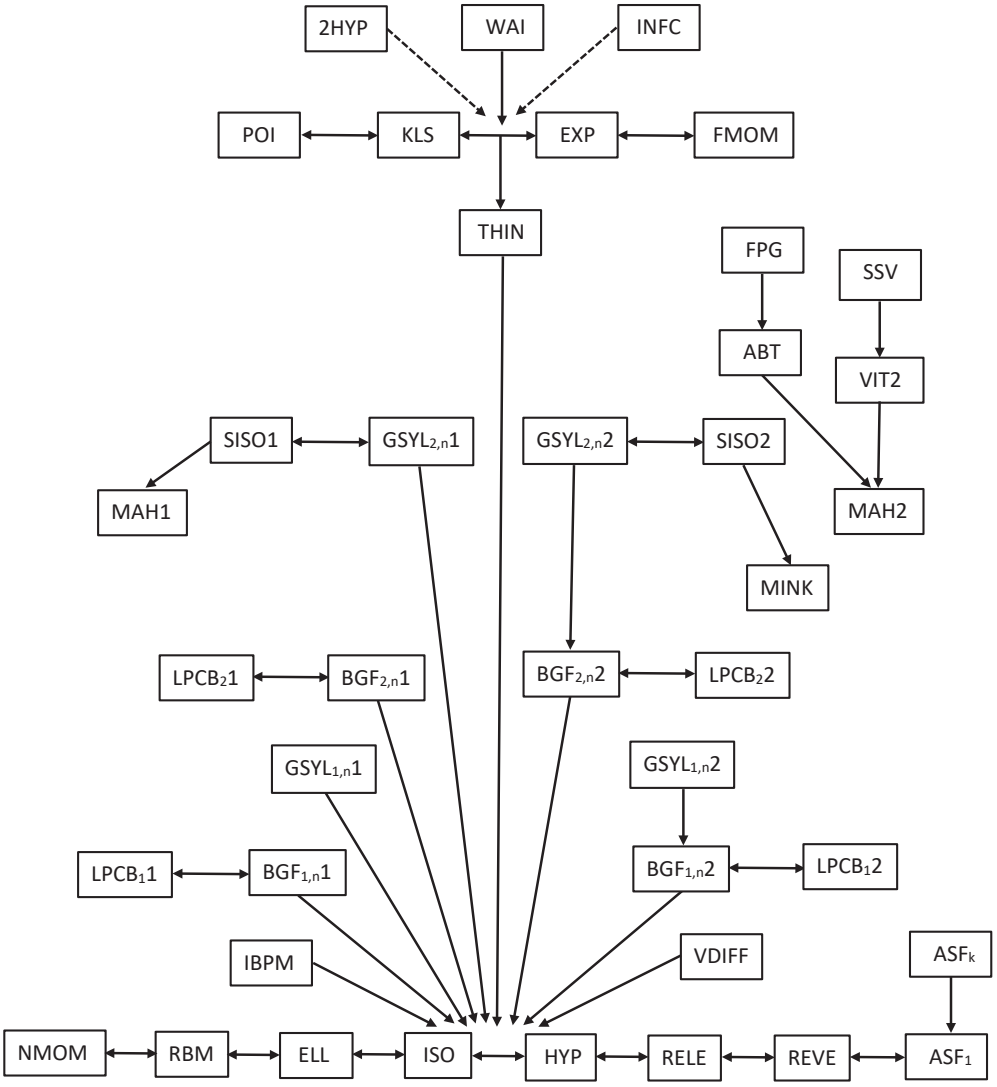

FIGURE 1. Relations between conjectures around the slicing problem or hyperplane conjecture (Problem 8.3) and Mahler's conjecture (Problem 9.2). Dotted arrows indicate that only a symmetric version of KLS is known to follow. All the conjectures on the lowest level are true.

Several more equivalent forms of HYP are known, set in the labyrinthine theory expounded in [16, Chapter 10] and [50], and surveyed in [147]. The following one was proposed by Dafnis and Paouris [66].

Negative moment conjecture (NMOM). There are universal constants $C, \xi > 0$ such that if $K \in \mathcal{K}_0^n$ is isotropic, then

$$\max\{p \geq 1 : M_2(K) \leq \xi M_{-p}(K)\} \geq C\, n,$$

where

$$M_p(K) = \left(\int_K \|x\|^p \, dx \right)^{1/p}.$$

Note that NMOM can be formulated in terms of dual volumes of K, since using (A.63), p. 412, we obtain

$$M_p(K)^p = \frac{n}{n+p} \widetilde{V}_{n+p}(K).$$

By [66, Theorem 6.5] (see also [50, Theorems 6.4.2 and 6.4.3]), we have

$$\mathsf{HYP} \Leftrightarrow \mathsf{NMOM}.$$

Two further equivalent statements come from information theory. If X is a random vector in $\mathbb{E}^n$ with probability density function f, its *differential entropy* $h(X) = h(f)$ is defined by (5). The notation $\mathrm{Ent}(X)$ is also often used.

Relative entropy conjecture (RELE). There is a universal constant $C > 0$ such that if X is a random vector in $\mathbb{E}^n$ whose probability density function f is log-concave, then

$$D(X) = D(f) = h(G) - h(X) \leq C\, n, \tag{6}$$

where G is a Gaussian random vector with the same covariance matrix as X.

Bobkov and Madiman [42] prove that

$$\mathsf{HYP} \Leftrightarrow \mathsf{RELE},$$

giving credit on [44, p. 3320] to Ball for the idea of such an equivalence. See also [43, Conjectures V.4 and V.5] for this and a further conjecture equivalent to RELE, and [45] for related results. Madiman, Nayar, and Tkocz [171, Section 3] note that the extremal functions f for (6) differ from those for ISO.

The *entropy power* of a random variable X is defined by $N(X) = (2\pi e)^{-1} e^{2h(X)/n}$. The famous entropy power inequality of Shannon (see [264book, (55), p. 384]) states that if X and Y are independent random vectors in $\mathbb{E}^n$ with probability densities in $L^p(\mathbb{E}^n)$ for some $p > 1$, then

$$N(X+Y) \geq N(X) + N(Y). \tag{7}$$

Reverse entropy power inequality conjecture (REVE). There is a universal constant $C > 0$ such that if $X_1, \ldots, X_k$ are i.i.d. random vectors in $\mathbb{E}^n$ whose probability density function is the density of an isotropic log-concave measure, then

$$N(X_1 + \cdots + X_k) \leq C\,(N(X_1) + \cdots + N(X_k)) = CkN(X_1).$$

Marsiglietti and Kostina [174, Theorem 1] (compare [42, Theorem 3]) prove that REVE $\Leftrightarrow$ RELE and hence

$$\mathsf{HYP} \Leftrightarrow \mathsf{REVE}.$$

Further versions of HYP continue to appear due to the fundamental significance of the isotropic constant. For example, Fresen [86] finds a reformulation in terms of the convex floating body, and Alonso-Gutiérrez and Brazitikos [12, Theorem 1.1] provide an equivalent statement in terms of the optimal constant $a(K)$, where $K \in \mathcal{K}_0^n$ is centered, for which there is an orthonormal basis $\{u_1, \dots, u_n\}$ of $\mathbb{E}^n$ such that the reverse dual Loomis–Whitney inequality

$$\lambda_n(K)^{n-1} \le a(K) \prod_{j=1}^{n} \lambda_{n-1}(K \cap u_j^\perp)$$

holds (compare Meyer's inequality in Note 9.7). Berndtsson, Mastrantonis, and Rubinstein [34] show that HYP would follow from their lower bounds for the L^p-Mahler volume (defined in the report below for Problem 9.2), if a certain convexity condition holds (the Ricci curvature of Bergman metrics of tube domains over a convex body has an upper bound independent of n). We mention in passing articles by Brazitikos and Liakopoulos [51] and Liakopoulos [168], where versions of HYP for the surface area and other quermassintegrals of central sections of convex bodies are considered.

Bourgain's $L_K \le Cn^{1/4} \log n$ bound for centered bodies K, mentioned in [88book, Note 8.9], is presented in [16, Theorem 10.3.3], [50, Section 3.3 and p. 135], and [92, Sections 2.4 and 2.5]. It was improved to $L_K \le Cn^{1/4}$ by Klartag [132], and this holds even for general bodies (again by [131, Remark 2, p. 392] or [200], see also [16, Theorem 10.5.4]). This was generalized and given an alternative proof by Klartag and Milman [146, Theorem 1.1] (see also [50, Theorems 7.3.2 and 7.5.15]). After Ball (see [25]) noticed that the Kannan–Lovász–Simonovits conjecture conjecture KLS $\Rightarrow$ HYP and Eldan and Klartag [75] proved that the thin shell conjecture THIN $\Rightarrow$ HYP (see below), these two still open conjectures became avenues for further analytical attacks on HYP. In particular, Eldan's technique of stochastic localization from [73] (see also [74] and the discussion below about progress on KLS) was employed by Lee and Vempala [164] to retrieve Klartag's bound via one for KLS. Chen [61] also used stochastic localization in his spectacular work on KLS (see the paragraph below on evidence for KLS and THIN). This led to the bound $L_K \le Cn^{o(1)}$, asymptotically better than any positive power of n. Subsequent work follows the same approach. A further significant improvement was made by Klartag and Lehec [142], who proved that $L_K \le C(\log n)^4$. The exponent was slightly lowered by Jambulapati, Lee, and Vempala [121], but soon after, another sizeable step, due again to Klartag [138], gave $L_K \le C\sqrt{\log n}$. Guan's remarkable paper [100] improved this to $L_K \le C \log\log n$. Much of it is devoted to proving [100, Lemma 2.1], a bound that allowed Klartag and Lehec [143] to deliver the death blow: HYP is true!

Needless to say, the proof of HYP is difficult. Guan's proof of [100, Lemma 2.1] alone is quite long and intricate. Klartag and Lehec [143] also employ Klartag's improved Lichnerowicz inequality mentioned just before (27) below; Milman's theory of M-ellipsoids (specifically, [50, (1.13.3), p. 57]); and stability results for the entropy power inequality (7) obtained by

Eldan and Mikulincer. Bizeul [36] finds an alternative to the approach of Klartag and Lehec, achieving this by proving a near-optimal small-ball estimate for isotropic random vectors X in $\mathbb{E}^n$, i.e., an upper bound for the probability that X concentrates around a point. His proof still uses stochastic localization and Guan's lemma [100, Lemma 2.1], however. Further simplifications will no doubt appear, but for now, we refer the reader seeking more details to online talks by Chen [197, January 5, 2021], Klartag [197, January 14, 2025] and Bizeul [197, January 28, 2025], notes by Klartag on Chen's work [137] and on Guan's [139], and a survey by Klartag and Lehec [144].

For the historical record, HYP was verified for various classes of convex bodies (meaning that for each class, L_K is bounded by a universal constant for each member K of the class), listed with references in [92, p. 65] and [153, p. 566], and often treated in [50], to which we shall refer when possible for details and further citations. As well as projection bodies (see also [50, Section 4.2.3]) and intersection bodies (and therefore polar projection bodies), already mentioned in Note 9.8, HYP is known to be true for unit balls of subspaces of L^p, for a fixed $0 \le p < \infty$ [23book], [617book]; unit balls of subspaces of quotients of L^p, $1 < p \le \infty$ [411book], [617book]; unconditional convex bodies [50, Section 4.1] and centered convex bodies within a given Banach–Mazur distance of them [50, Corollary 4.2.6(ii)]; centered convex bodies with outer volume ratio bounded above by a fixed constant [50, Proposition 4.2.1]; unit balls of 2-convex spaces with fixed constant [50, Section 4.2.2]; unit balls of the Schatten classes [50, Section 4.3]; convex polytopes with no more than a given number of vertices or facets [50, Section 4.4]; and k-intersection bodies [152]. Klartag and Kozma [141] (see also [50, Section 11.4]) verified the conjectures for certain random polytopes; more specifically, they proved that if K is the convex hull of $m \ge n$ independent standard Gaussian vectors in $\mathbb{E}^n$, then $L_K < C$ with probability at least $1 - Ce^{-cn}$, where c and C are universal constants (also not depending on m). This was followed by a series of related works, to which references can be found in one of the most recent, that by Prochno, Thäle, and Turchi [205] (see also [50, Sections 11.5 and 11.6]). In some cases, the classes of bodies mentioned in this paragraph also satisfy certain of the potentially stronger conjectures in Figure 1 or represent positive partial answers to the measure slicing problem, as described below.

Before turning to other conjectures stronger than HYP, we mention a discrete version of HYP and also explain how HYP bears on some other well-known open questions. The discrete version was posed by Koldobsky at an AIM meeting in 2013. It asks for the best constant c_n such that for every centered convex body in $\mathbb{E}^n$ with $\dim(K \cap \mathbb{Z}^n) = n$, there is an $S \in \mathcal{G}(n, n-1)$ such that

$$|K \cap \mathbb{Z}^n| \le c_n |K \cap \mathbb{Z}^n \cap S| \, \lambda_n(K)^{1/n}.$$

Currently the best estimate, due to Freyer and Henk [88], is $c_n \le C_1 n^{10/3} (\log n)^{C_2}$, where C_1 and C_2 are universal constants. See also the discussion in [96, Section 8].

HYP has consequences for *Hadwiger's conjecture* from combinatorial geometry posed in 1957, which states that at most 2^n translates of the interior of a convex body K in $\mathbb{E}^n$ are needed to cover K itself, the upper bound 2^n only required if K is a parallelepiped. It also appertains to *Ehrhart's volume conjecture* from 1964, that if K is a convex body in $\mathbb{E}^n$ with

centroid at o and such that $\mathbb{Z}^n \cap \operatorname{int} K = \{o\}$, then $\lambda_n(K) \le (n+1)^n/n!$, with equality if K is the simplex $K = (n+1)\operatorname{conv}\{e_1, \dots, e_n\} - (1, 1, \dots, 1)$. Both conjectures have been confirmed when $n = 2$ but are open for $n \ge 3$. Until recently, the best known general bounds for the two conjectures were

$$\binom{2n}{n}(n \log n + n \log\log n + 5n) = O\left(4^n \sqrt{n} \log n\right) \quad \text{and} \quad 4^n,$$

respectively. Huang, Slomka, Tkocz, and Vritsiou [112] improved these bounds by utilizing thin-shell estimates for log concave measures from [101]. Campos, van Hintum, Morris, and Tiba [55] follow their approach but instead use Klartag and Lehec's bound for HYP from [142] to do even better, obtaining the bound $4^n \exp(-Cn/(\log n)^8)$ for both conjectures. Since HYP is true, we now have the current best bound of $4^n \exp(-Cn)$ instead. In fact, these estimates have a third ingredient in common, which is to obtain a good lower bound for $\lambda_n(K \cap (-K))/\lambda_n(K)$; the minimum is thought to be attained by a simplex, yielding a lower bound of the order of $(2/e)^n$, but this has not been proved.

Sharp isotropic constant conjectures. If $K \in \mathcal{K}_0^n$, then $L_K \le L_{\triangle_n}$ (SISO1), where $\triangle_n$ is a regular simplex with centroid at the origin and unit volume. If K is also centered, then $L_K \le L_{C_n}$ (SISO2), where C_n is a centered unit cube.

SISO2 is explicitly stated in [596book, p. 312], but Ball's suggestion in [23, p. 85] (see also [20book, p. 83]) is apparently its first appearance in print.

Computations show that

$$L_{\triangle_n} = \frac{(n!)^{1/n}}{(n+1)^{(n+1)/(2n)}\sqrt{n+2}} \quad \text{and} \quad L_{C_n} = 1/\sqrt{12}; \tag{8}$$

the former is shown in [180, p. 88], while the latter is easily checked. Since $L_{\triangle_n}$ is uniformly bounded in n, we have

$$\text{SISO1} \Rightarrow \text{HYP} \quad \text{and} \quad \text{SISO2} \Rightarrow \text{HYP}.$$

Of course, one can also ask to identify the bodies that achieve equality in the conjectured inequalities. Affirmative answers to SISO1 and SISO2 with the expected equality conditions are available when $n = 2$, via those to related conjectures described below. Building on work of Campi, Colesanti, and Gronchi [54], Meyer and Reisner [183] prove that for $n \ge 2$, any local maximizer of L_K, for either SISO1 or SISO2, cannot be C_+^2 at any point of ∂K, while Rademacher [207, (2), p. 309] shows that if a maximizer of L_K for SISO1 is a simplicial polytope, then it must be a simplex.

Fradelizi and Marín Sola [81] propose the following functional versions of the sharp isotropic constant conjectures. Let

$$f_0(x) = \exp(-\Sigma_{i=1}^n x_i) 1_{[-1,\infty)}(x), \ f_1(x) = \exp(-\Sigma_{i=1}^n |x_i|), \text{ and } f_\infty(x) = 1_{[-1,1]^n}(x).$$

They conjecture that if f is a nonnegative log-concave function on $\mathbb{E}^n$ with nonzero integral, then $L_f \le L_{f_0} = 1$ and $\widehat{L}_f \le \widehat{L}_{f_0} = e^{-1}$. Denote these statements by $P(n)$ and $\widehat{P}(n)$, respectively. They further conjecture that if f is also even, then $L_f \le L_{f_1} = 1/\sqrt{2}$ and

$\widehat{L}_f \le \widehat{L}_{f_\infty} = 1/\sqrt{12}$. Denote these statements by $Q(n)$ and $\widehat{Q}(n)$, respectively. All four statements are true when $n = 1$; see the references provided in [81]. It is proved in [81] that if $\widehat{P}(n+1)$ holds, then $\mathsf{SISO1}$ holds in $\mathbb{E}^n$, and conversely, if $\mathsf{SISO1}$ holds in $\mathbb{E}^{n+m}$ for all $m \in \mathbb{N}$, then $\widehat{P}(n)$ holds. It is also shown that if $\widehat{Q}(n)$ is true, then $\mathsf{SISO2}$ holds in $\mathbb{E}^n$, and conversely, if $\mathsf{SISO2}$ holds in $\mathbb{E}^{n+m}$ for all $m \in \mathbb{N}$, then $\widehat{Q}(n)$ is true.

In the context of convex bodies, all the above conjectures are inextricably linked with moments of volumes of random convex polytopes contained in a convex body. To explain this, for a bounded Borel set A in $\mathbb{E}^n$, $k \ge n$, and $p > 0$, define

$$h_{p,k}(A) = \int_A \cdots \int_A \lambda_n(\operatorname{conv}\{x_0, x_1, \dots, x_k\})^p \, dx_0 \cdots dx_k$$

and, when A is centered,

$$j_{p,k}(A) = \int_A \cdots \int_A \lambda_n(\operatorname{conv}\{\pm x_1, \dots, \pm x_k\})^p \, dx_1 \cdots dx_k.$$

These quantities often appear in normalized form or modified as a pth mean, and with different notation. Their relevance stems from the formulas

$$L_K = \left(\frac{n!\, h_{2,n}(K)}{(n+1)\lambda_n(K)^{n+3}} \right)^{1/2n} \quad \text{and} \quad L_K = \frac{1}{2} \left(\frac{n!\, j_{2,n}(K)}{\lambda_n(K)^{n+2}} \right)^{1/2n}, \tag{9}$$

for $K \in \mathcal{K}_0^n$ and centered $K \in \mathcal{K}_0^n$, respectively. These are stated by Meckes [180, p. 88], [596book, p. 312], the latter providing references. For example, Ball [20book] utilizes $j_{2,n}(K)$. From (9) and Stirling's formula, we see that ISO, and hence HYP, is equivalent to $h_{p,n}(K)^{1/np} \le C/\sqrt{n}$ or $j_{p,n}(K)^{1/np} \le C/\sqrt{n}$ when $\lambda_n(K) = 1$ and $p = 2$, but Meckes [180, p. 88] observes that this is true for any $p \ge 1$.

Slightly adapting the terminology of Meckes [596book], we shall refer to the problem of finding, for each $n \ge 2$, $k \ge n$, and $p \ge 1$, those convex bodies (or centered convex bodies) K for which $h_{p,k}(K)$ (or $j_{p,k}(K)$, respectively) achieve their extremal values, as the *generalized Sylvester problem*. The case $k = n = 2$, $p = 1$ arose from a discussion in the *Educational Times* of 1864–5 about the earlier version of Sylvester's problem stated in Note 9.4. Pfiefer [204] provides a historical account. It is known that the lower bounds are attained precisely by ellipsoids; the corresponding inequality $h_{p,k}(K) \ge h_{p,k}(B^n)$, where $\lambda_n(K) = \kappa_n$, is the *Blaschke–Groemer inequality* (see [227, Theorem 10.3.4]), and Meckes [180, Theorem 1(2)] provides the corresponding inequality for $j_{p,k}(K)$.

Generalized Sylvester conjectures. If $K \in \mathcal{K}_0^n$ has unit volume, $k \ge n$, and $p \ge 1$, then $h_{p,k}(K) \le h_{p,k}(\triangle_n)$ ($\mathsf{GSYL}_{p,k}1$) and if K is also centered, then $j_{p,k}(K) \le j_{p,k}(C_n)$ ($\mathsf{GSYL}_{p,k}2$).

The special case $\mathsf{GSYL}_{1,n}1$ has also been called the *simplex conjecture*. An interesting result of Bárány and Buchta [28] in the same direction shows that for each $K \in \mathcal{K}_0^n$ with $\lambda_n(K) = 1$, there is an $N = N(K) \ge n$ such that $h_{1,k}(K) \le h_{1,k}(\triangle_n)$ for all $k \ge N$.

Again, it can be asked to find the bodies that achieve equality in the conjectured inequalities. From (9), it is evident that

$$\mathsf{GSYL}_{2,n}1 \Leftrightarrow \mathsf{SISO1} \quad \text{and} \quad \mathsf{GSYL}_{2,n}2 \Leftrightarrow \mathsf{SISO2}.$$

An affirmative answer to $\mathsf{GSYL}_{p,k}1$ in $\mathbb{E}^2$ was provided by Campi, Colesanti, and Gronchi [54], generalizing earlier work of Blaschke (see [50, p. 135] and [227, pp. 540–1]) and Dalla and Larman [68]. Saroglou [222] proves that when $k = n = 2$ and $p \geq 1$, simplices are the only maximizers, extending the result for $p = 1$ of Giannopoulos [91]. Campi, Colesanti, and Gronchi [54] employ shadow systems, by means of which they also show that maximizers cannot be too smooth; specifically, their boundaries cannot be of class C^2_+ in a nonempty relatively open subset (see also [50, Theorems 3.2 and 3.3]). Meckes [596book, Theorem 9] adapts the tools developed in [54] to obtain analogous results for centered bodies, in particular an affirmative answer to $\mathsf{GSYL}_{p,k}2$ in $\mathbb{E}^2$.

For a bounded Borel set A in $\mathbb{E}^n$, $k \geq n$, and $p > 0$, define

$$g_{p,k}(A) = \int_A \cdots \int_A \lambda_n(\operatorname{conv}\{o, x_1, \ldots, x_k\})^p \, dx_1 \cdots dx_k.$$

Compare the set function $g_{m,k}$ defined on p. 353, which features in the Busemann random simplex inequality, Theorem 9.2.6; here we are abusing notation, but $g_1 = g_{1,n}$ is the same in both cases. The lower bound for $g_{p,k}$ on convex bodies is attained precisely for centered ellipsoids, by the *Busemann–Groemer inequality* [227, Theorem 10.3.5].

Busemann–Groemer functional conjectures. If K is a convex body of unit volume with centroid at the origin in $\mathbb{E}^n$, $k \geq n$, and $p \geq 1$, then $g_{p,k}(K) \leq g_{p,k}(\triangle_n)$ ($\mathsf{BGF}_{p,k}1$). If K is also centered, then $g_{p,k}(K) \leq g_{p,k}(C_n)$ ($\mathsf{BGF}_{p,k}2$).

In [50, Proposition 3.5.2], it is shown that if $p \geq 1$ and $K \in \mathcal{K}^n_0$ is centered and of volume 1, then

$$g_{p,n}(K) \leq h_{p,n}(K) \leq (n+1)^p g_{p,n}(K). \tag{10}$$

The left-hand inequality in (10) yields the relation

$$\mathsf{GSYL}_{p,n}2 \Rightarrow \mathsf{BGF}_{p,n}2.$$

Moreover, [50, Corollary 3.5.8] uses (10) to conclude that if the simplex conjecture is true, the hyperplane conjecture follows, i.e.,

$$\mathsf{GSYL}_{1,n}1 \Rightarrow \mathsf{HYP}.$$

The previous implication and (10) are due to Giannopoulos and first appeared in his 1993 PhD thesis.

We shall continue to consider the Busemann–Groemer functional conjectures together with the following related ones, in which $\Gamma_p K$ denotes the centroid body of K, defined in Note 9.5.

L_p-centroid body conjectures. If K is a convex body of unit volume with centroid at the origin in $\mathbb{E}^n$ and $p \geq 1$, then $\lambda_n(\Gamma_p K) \leq \lambda_n(\Gamma_p \triangle_n)$ (LPCB_p1). If K is also centered, then $\lambda_n(\Gamma_p K) \leq \lambda_n(\Gamma_p C_n)$ (LPCB_p2).

We adopt the usual custom of writing ΓK for $\Gamma_1 K$. A connection to the hyperplane conjecture is explicitly noted by Milman and Pajor [621book, Proposition 5.4], who show that HYP is equivalent to proving that $\lambda_n(\Gamma K)^{1/n} \le C/\sqrt{n}$ when K is centered and $\lambda_n(K) = 1$ or to proving that $\lambda_n(\Gamma_2 K) \le C$ when K is centered and $\lambda_n(K) = 1$. For $p = 1, 2$, it is known that when $\lambda_n(K) = 1$, there are constants $c_{p,n}$ depending only on n and p such that $\lambda_n(\Gamma_p K) = c_{p,n}\, g_{p,n}(K)^{1/p}$. (See [148book, p. 133]; for $p = 1$, this is due to Petty (see Theorem 9.1.5 and Note 9.1), while the case $p = 2$ goes back to Blaschke; for $p \ne 1, 2$, such formulas are apparently not available.) This leads to yet further statements equivalent to HYP, involving $g_{1,n}(K)$ and $g_{2,n}(K)$. From these facts, it follows that for $p = 1$ or 2,

$$\mathsf{BGF}_{p,n}1 \Leftrightarrow \mathsf{LPCB}_p 1 \Rightarrow \mathsf{HYP} \quad \text{and} \quad \mathsf{BGF}_{p,n}2 \Leftrightarrow \mathsf{LPCB}_p 2 \Rightarrow \mathsf{HYP}.$$

It is expected that the upper bound in each set of conjectures is attained precisely for simplices or parallelotopes, respectively. With this equality condition, $\mathsf{LPCB}_p 2$ was proposed for $p = 1, 2$ by Bisztriczky and Böröczky [67book, Conjectures 1.2 and 2.2], who proved both cases, and hence $\mathsf{BGF}_{p,k}2$ for $p = 1, 2$, when $n = 2$. (This is already partially mentioned in Note 9.4.) They also proposed a version of $\mathsf{LPCB}_p 1$ for $p = 1, 2$, with the weaker restriction $o \in K$, and proved that the corresponding inequalities hold when $n = 2$, with equality if and only if K is a triangle with o as a vertex. As was mentioned in Note 9.5, Campi and Gronchi [147book] extend these results by means of shadow systems; their results are more general, but they include proofs of $\mathsf{LPCB}_p 1$ and $\mathsf{LPCB}_p 2$, for all $p \ge 1$ when $n = 2$, and hence $\mathsf{BGF}_{p,n}1$ and $\mathsf{BGF}_{p,n}2$ for $n = 2$ and $p = 1, 2$, with equality conditions.

Problem 8.3 as stated is now often called the *isomorphic Busemann–Petty problem* and its equivalence to HYP was explained in Note 9.8. Other variants of the Busemann–Petty problem are not relevant to HYP and will not be discussed here, but some, leading to generalizations of HYP, originate in the ideas of Zvavitch [872book] (see also [465book, Section 5.4]) to consider the Busemann–Petty problem for arbitrary measures and of Koldobsky [149] to obtain stability versions. For a brief description, we first note that the version of HYP in (9.10), p. 385 is equivalent to asking if there is a universal constant C such that for $1 \le k \le n-1$ and centered $K \in \mathcal{K}_0^n$, we have

$$\lambda_n(K)^{(n-k)/n} \le C^k \max_{S \in \mathcal{G}(n,n-k)} \lambda_{n-k}(K \cap S).$$

Indeed, the previous inequality results from iterating (9.10), p. 385, which is the case $k = 1$. In [151, Problem 1], Koldobsky asks if there is a universal constant C such that for any $1 \le k \le n-1$, any centered $K \in \mathcal{K}_0^n$, and any even measure μ in $\mathbb{E}^n$ with nonnegative continuous density function f, we have

$$\mu(K) \le C^k \max_{S \in \mathcal{G}(n,n-k)} \mu^S(K \cap S)\, \lambda_n(K)^{k/n}, \tag{11}$$

where μ^S is the measure in S with density f with respect to λ_{n-k}. One might call this the *measure slicing problem*. By taking $f \equiv 1$, we see that an affirmative answer would also dispose of HYP. Koldobsky [150] shows that (11) holds with a factor $n^{k/2}$ on the right-hand side. (Chasapis, Giannopoulos, and Liakopoulos [58] demonstrate that this remains true for arbitrary $K \in \mathcal{K}_0^n$ with $o \in \operatorname{int} K$ when f is only assumed to be nonnegative and locally

integrable.) Koldobsky [151] proves that for each $\alpha \in (0,1)$, there is a $C = C(\alpha)$ such that (11) is true whenever $k \geq \alpha\, n$. Extending a result of Milman [617book, Corollary 5.4], Koldobsky [151, Corollary 1] also obtains (11) with K replaced by a centered star body L and with a factor $d_{ovr}(L, \mathcal{BP}_k^n)$ on the right-hand side. The latter quantity is an *outer volume ratio distance*, defined for a star body L and class $\mathcal{E}$ of star bodies in $\mathbb{E}^n$ by

$$d_{ovr}(L,\mathcal{E}) = \inf\left\{\left(\frac{\lambda_n(M)}{\lambda_n(L)}\right)^{1/n} : L \subset M,\ M \in \mathcal{E}\right\},$$

and $\mathcal{BP}_k^n$ is the class of $(k, n-k)$-intersection bodies in $\mathbb{E}^n$, defined in Note 8.7 on p. 341, where it was noted that $\mathcal{BP}_1^n = \mathcal{I}_n$. In a survey, Koldobsky [153, p. 567] lists many other positive results concerning the measure slicing problem. However, it turns out that (11) is generally false, since Klartag and Livshyts [145], improving on an earlier construction of Klartag and Koldobsky [140], find an example showing that up to a universal constant, a factor $\sqrt{n}$ must be inserted on the right-hand side of (11) for the inequality to hold generally when $k = 1$. Further results on the measure slicing problem may be found in [40], [156], and [236].

Despite the falsity of (11), the following conjecture, proposed as a problem in [157, p. 262], is open, where we use notation from the previous paragraph.

Isomorphic Busemann–Petty conjecture for measures (IBPM). Let K and L be centered convex bodies in $\mathbb{E}^n$ and let μ be an even measure in $\mathbb{E}^n$ with a nonnegative continuous density function f. There is a universal constant C such that if $\mu^{u^\perp}(K \cap u^\perp) \leq \mu^{u^\perp}(L \cap u^\perp)$ for all $u \in S^{n-1}$, then $\mu(K) \leq C\mu(L)$.

Here $\mu^{u^\perp}$ is the measure in $u^\perp$ with density f with respect to λ_{n-1}. Taking $f \equiv 1$, and bearing in mind the equivalence of the slicing and isomorphic Busemann–Petty problems, we see that

$$\text{IBPM} \Rightarrow \text{HYP}.$$

Koldobsky and Zvavitch [157, Theorem 1] prove that the hypothesis of IBPM yields the conclusion $\mu(K) \leq d_{BM}(K, \mathcal{I}_n)\, \mu(L)$, where $d_{BM}(K, \mathcal{I}_n)$ is the Banach–Mazur distance from K to the class $\mathcal{I}_n$ of intersection bodies in $\mathbb{E}^n$, and from this, the fact that $B^n \in \mathcal{I}_n$, and Theorem 4.2.12 (John's theorem), that the weaker conclusion $\mu(K) \leq \sqrt{n}\, \mu(L)$ is valid. Unlike the measure slicing problem, is not known whether $\sqrt{n}$ is optimal, nor whether $d_{BM}(K, \mathcal{I}_n)$ can be replaced by $d_{ovr}(K, \mathcal{I}_n)$, which would give the desired conclusion for unconditional convex bodies. Of course, a version of IBPM can be formulated for lower-dimensional sections, and this is considered by Chasapis, Giannopoulos, and Liakopoulos [58] and Giannopoulos and Koldobsky [93].

Thus we have seen that the measure slicing problem and IBPM may be fundamentally different from each other, unlike their counterparts in the special case when $\mu = \lambda_n$. However, Koldobsky, Paouris, and Zvavitch [154] find a common approach by assuming that $\mu_1^S(K \cap S) \leq \mu_2^S(L \cap S)$, for all $S \in \mathcal{G}(n, n-k)$, where μ_1 and μ_2 are measures with possibly different density functions f_1 and f_2, and seeking an upper bound for $\mu_1(K)$ in terms of $\mu_2(L)$. In this way, they are able to generalize several results from [40], [140], [145], and [151]. One could of course state stronger forms of IBPM involving two measures and/or lower-dimensional sections; we shall

not do so here, but instead refer to Giannopoulos, Koldobsky, and Zvavitch [95] for the most general results in this direction. Still further variations on the theme of Busemann-Petty-type comparison theorems and slicing questions for functions are investigated by Giannopoulos, Koldobsky, and Zvavitch [96], Haddad and Koldobsky [104], and Koldobsky, Roysdon, and Zvavitch [155].

If L is a star body in $\mathbb{E}^n$ and $S \in \mathcal{G}(n, n-k)$, where $0 \le k \le n-2$, define the *average section functional* $\operatorname{as}(L \cap S)$ by

$$\operatorname{as}(L \cap S) = \int_{S^{n-1} \cap S} \lambda_{n-k-1}(L \cap S \cap u^{\perp})\, du.$$

Note that $\operatorname{as}(L) = \operatorname{as}(L \cap \mathbb{E}^n)$ corresponds to the case when $k = 0$. Also, up to a constant, the average section functional is a dual volume, since

$$\operatorname{as}(L \cap S) = (n-k)\kappa_{n-k-1}\widetilde{V}_{n-k}(L \cap S),$$

as can be seen by using (A.58), p. 410 (with n replaced by $n-k$ and $n-k-1$) and Kubota's formula, Theorem A.2.7, and taking $L = B^n$ to evaluate the constant. The following conjectures were posed as a question by Brazitikos, Dann, Giannopoulos, and Koldobsky [49, Question 1.1].

Average section functional conjectures (ASF_k). If $1 \le k \le n-2$, there is a universal constant C such that if $K \in \mathcal{K}_0^n$ is centered, then

$$\operatorname{as}(K) \le C^k \lambda_n(K)^{k/n} \max_{S \in \mathcal{G}(n,n-k)} \operatorname{as}(K \cap S). \tag{12}$$

In [49, Proposition 4.5], it is shown that

$$\mathsf{ASF}_1 \Leftrightarrow \mathsf{HYP}.$$

From earlier work of Koldobsky cited in [49] it is known that (12) holds in the class of intersection bodies. In [49, Theorem 1.3], it is proved that (12) is true, even for centered star bodies, with an extra factor $d_{ovr}(L, \mathcal{BP}_k^n)$ on the right-hand side, yielding the conjecture for other classes besides. The authors also prove in [49, Theorem 1.6] that (12) holds with an extra factor L_K^k on the right-hand side, as well as several other related results.

A different strengthening of HYP is considered by Giannopoulos and Koldobsky [94, Question 1.1].

Volume difference conjecture (VDIFF). There is a universal constant C such that if $1 \le k \le n-1$ and $\gamma_{k,n} = \min \gamma > 0$ satisfies

$$\lambda_n(K)^{(n-k)/n} - \lambda_n(L)^{(n-k)/n} \le \gamma^k \max_{S \in \mathcal{G}(n,n-k)} (\lambda_{n-k}(K \cap S) - \lambda_{n-k}(L \cap S)),$$

for all centered $K, L \in \mathcal{K}_0^n$ with $L \subset K$, then $\sup_{n,k} \gamma_{n,k} \le C$.

Taking $k = 1$, $L = rB^n$, and letting $r \to 0$ yields inequality (9.10) on p. 385. In other words,

$$\mathsf{VDIFF} \Rightarrow \mathsf{HYP}.$$

Along with several other related results, it is shown in [94, Corollary 1.3] that the previous inequality holds with an extra factor of $(\sqrt{n/k}\,(\log(e\,n/k))^{3/2})^k$ on the right-hand side. Extensions to arbitrary measures are discussed in [96, Section 7].

A remarkable result of Klartag [136] establishes a direct connection between Problems 8.3 and 9.2. In [136, Theorem 1.1], he proves that if a convex body K in $\mathbb{E}^n$ contains the origin in its interior and is a local minimizer of the volume product $v(K) = \lambda_n(K)\lambda_n(K^*)$, then

$$L_K L_{K^*}\, v(K)^{1/n} \geq \frac{1}{n+2}.$$

Consequently, SISO1 and (8) imply that

$$\frac{1}{(n+2)^n} \leq L_K^n L_{K^*}^n\, v(K) \leq L_{\triangle_n}^{2n} v(K) = \frac{(n!)^2}{(n+1)^{n+1}(n+2)^n} v(K)$$

and hence $v(K) \geq (n+1)^{n+1}/(n!)^2 = v(\triangle_n)$. Thus

$$\mathsf{SISO1} \Rightarrow \mathsf{MAH1},$$

the Mahler conjecture for bodies containing the origin in their interiors (see the report for Problem 9.2). A shorter proof of this result has been offered by Balacheff, Solanes, and Tzanev [21], and Fradelizi and Marín Sola [81] establish a functional version of it; see the report for Problem 9.2 below.

We have seen that (11) is generally false, so despite Figure 1, it is not true that any reasonably plausible conjecture stronger than HYP is open at the present time. Klartag [136, Section 5] also offers a cautionary note by constructing unconditional convex bodies that form counterexamples to [161, Conjecture 5.1]. The latter proposes that the maximum of

$$\phi(K) = \frac{1}{v(K)} \int_K \int_{K^*} (x \cdot y)^2\, dx\, dy,$$

over all centered convex bodies in $\mathbb{E}^n$, where $v(K)$ is the Mahler volume, is attained when K is an ellipsoid. Alonso-Gutiérrez [8] (see also [50, p. 137], [161, p. 890], and [227, pp. 577–8]) showed that this would imply both HYP and the Blaschke–Santaló inequality for centered convex bodies, Theorem 9.2.11, and moreover verified it for l_p^n balls.

The following conjecture is ascribed to Minkowski although apparently not published by him. It has important consequences in number theory, such as [179, Conjecture 1.2], and is known to be true when $n \leq 10$; see [128], [228], and the references given there. Note that the stronger Woods' conjecture, also proved for $n \leq 10$ in [128], is now known to be false when $n \geq 24$.

Minkowski's conjecture (MINK). Let $o \in \Lambda$ be an n-dimensional lattice in $\mathbb{E}^n$ whose Voronoi cell at o has unit volume. Then for each $x \in \mathbb{E}^n$, there is a $v = (v_1, \ldots, v_n) \in \Lambda + x$ such that $v_1 v_2 \cdots v_n \leq 2^{-n}$.

Building on earlier results and the known connection [210, p. 592] between the two conjectures, Magazinov [172] proves that

$$\mathsf{SISO2} \Rightarrow \mathsf{MINK}.$$

To describe still further conjectures related to HYP, some terminology is required. Let μ be a finite Borel measure in $\mathbb{E}^n$. If A is a Borel set in $\mathbb{E}^n$, its (lower) *outer Minkowski content with respect to* μ is defined by

$$\mu^+(A) = \liminf_{\varepsilon \to 0+} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon},$$

where $A_\varepsilon = \{x : d(x, A) < \varepsilon\}$. If μ is an isotropic log-concave measure in $\mathbb{E}^n$,

$$\mathrm{Is}_\mu = \inf_A \frac{\mu^+(A)}{\min\{\mu(A), 1 - \mu(A)\}} = \inf_{\{A:\ \mu(A) \le 1/2\}} \frac{\mu^+(A)}{\mu(A)} = 2 \inf_{\{A:\ \mu(A) = 1/2\}} \mu^+(A) \tag{13}$$

is the *Cheeger constant* of μ, where A represents any Borel set in $\mathbb{E}^n$. The term derives from Cheeger's lower bound [60] $\lambda_1 \ge h^2/4$ for the least nonzero eigenvalue λ_1 of the Laplace–Beltrami operator on a Riemannian manifold in terms of its Cheeger constant h, defined in a way corresponding to (13). The right-hand equality in (13) is a consequence of the concavity of the *isoperimetric profile* $I(\mu)$ of μ, the pointwise maximal function $I : [0, 1] \to [0, \infty)$ such that $\mu^+(A) \ge I(\mu(A))$ for all Borel sets A in $\mathbb{E}^n$, proved in suitably general form by Milman (see [184, Theorem 1.8] and the remarks that follow). In the special case when μ is the normalized restriction of volume λ_n to a convex body K in $\mathbb{E}^n$, Jerison [122, p. 731] states that the infimum is achieved by an open set that attains

$$\min\left\{\mathcal{H}^{n-1}((\partial A) \cap \operatorname{int} K) :\ A \subset K \text{ is open and } \lambda_n(A) = \lambda_n(K)/2.\right\} \tag{14}$$

(The part of ∂A contained in ∂K is not counted by μ^+ since μ vanishes outside K.)

Several variants of these concepts are found in the swampy literature surrounding them. Jerison works with Borel subsets A of a domain Ω in $\mathbb{E}^n$ and calls a set $E \subset \Omega$ *isoperimetric in* Ω if it attains

$$\inf\{P(A)/\lambda_n(A) : \lambda_n(A) = \alpha\, \lambda_n(\Omega)\}, \tag{15}$$

where $0 < \alpha < 1$ is fixed and $P(A)$ is the perimeter of A. The case $\alpha = 1/2$ is equivalent to (14). In [122, Conjecture 2.3], he conjectures that when Ω is convex, the boundaries in Ω of open isoperimetric sets in Ω are Lipschitz graphs. Elsewhere, such as [52] and [167], sets attaining the infimum in (15) with the ratio $\lambda_n(A)/\lambda_n(\Omega)$ unrestricted are called *Cheeger sets*, and the infimum itself the Cheeger constant of Ω, a quite different usage. The endeavor of finding, computing, and classifying Cheeger sets overlaps with the study of isoperimetric problems restricted to certain regions or spaces, surveyed by Ros [213].

The following conjecture originates in the work of Kannan, Lovász, and Simonovits [125] and is expounded at length in [9] and [50, Chapter 14] (see also [10] and [165] for short surveys).

Kannan–Lovász–Simonovits conjecture (KLS). There is a universal constant $C > 0$ such that

$$\mathrm{Is}_n = \min\{\mathrm{Is}_\mu :\ \mu \text{ is an isotropic log-concave measure in } \mathbb{E}^n\} \ge C. \tag{16}$$

(Recently, authors have often preferred to work with $\psi_\mu = 1/\mathrm{Is}_\mu$ and $\psi_n = 1/\mathrm{Is}_n$ instead.)

KLS was motivated by various algorithmic problems, for example, computing the volume of a convex body; see, for example, [9, Section 1.2] and [165]. (Efficient algorithms that avoid

appeal to KLS have been proposed by Cousins and Vempala [63].) This application derives from the following alternate form of the conjecture (see [73, p. 532] or [159, (1.2), p. 3759]): There is a universal constant $C > 0$ such that for $0 < \alpha \le 1/2$ and a Borel subset A of a convex body K in $\mathbb{E}^n$ with centroid at o such that $\lambda_n(A) = \alpha\,\lambda_n(K)$, we have

$$\mathcal{H}^{n-1}((\partial A) \cap \operatorname{int} K) \ge \alpha\, C \inf_{u \in S^{n-1}} \lambda_{n-1}(K \cap u^\perp). \tag{17}$$

Taking $\alpha = 1/2$, this says that, up to a universal constant, the most efficient way to halve the volume of a convex body is by cutting it with a hyperplane. The algorithmic applications stem from the use of this fact in repeated bisections. Unfortunately, the equivalence of the formulation in (17), and the fact that it suffices to prove KLS when μ is the normalized restriction of volume λ_n to a convex body in $\mathbb{E}^n$, appear to be folklore shared by a few experts, since no proofs have appeared in print.

As we shall explain, KLS sits in the center of a heap of conjectures, all at least as strong as HYP. We now list a few of these known to be equivalent to KLS, sticking to our Euclidean setting even though extensions to Riemannian manifolds have been considered.

Poincaré inequality conjecture (POI). There is a universal constant $C > 0$ such that for every isotropic log-concave measure μ in $\mathbb{E}^n$ and every smooth function φ such that $\int_{\mathbb{E}^n} \varphi\, d\mu = 0$, one has

$$C \int_{\mathbb{E}^n} \varphi^2\, d\mu \le \int_{\mathbb{E}^n} \|\nabla \varphi\|^2\, d\mu. \tag{18}$$

Exponential concentration conjecture (EXP). There is a universal constant $C > 0$ such that if μ is an isotropic log-concave measure in $\mathbb{E}^n$ and g is a 1-Lipschitz function on $\mathbb{E}^n$, then

$$\mu\left(\{x : |g(x) - \mathbb{E}_\mu g| \ge t\}\right) \le e^{1-Ct}, \tag{19}$$

where $\mathbb{E}_\mu$ denotes expectation with respect to μ.

First moment concentration conjecture (FMOM). There is a universal constant $C > 0$ such that if μ is an isotropic log-concave measure in $\mathbb{E}^n$ and g is a 1-Lipschitz function on $\mathbb{E}^n$, then

$$C \int |g(x) - \mathbb{E}_\mu g|\, d\mu \le 1, \tag{20}$$

where $\mathbb{E}_\mu$ denotes expectation with respect to μ.

Thin shell conjecture (THIN). There is a universal constant $C > 0$ such that if μ is an isotropic log-concave measure in $\mathbb{E}^n$, then

$$\sigma_n = \left(\int_{\mathbb{E}^n} (\|x\| - \sqrt{n})^2\, d\mu(x) \right)^{1/2} \le C. \tag{21}$$

THIN is sometimes called the *variance conjecture*, because under its hypotheses, (21) is equivalent to $\operatorname{Var}_\mu \|X\|^2 \le Cn$, where X is a random variable whose probability density function is the density of μ; see [50, Lemma 12.3.1] or [75, Lemma 1.4]. (Perhaps for this reason, the constant σ_n has been defined differently as $\sup_\mu \sigma_\mu$, where $n\sigma_\mu^2 = \operatorname{Var}_\mu \|X\|^2$. In the context of THIN, the two definitions are equivalent in view of [50, Lemma 12.3.1] or [75,

Lemma 1.4].) The question is explicitly stated by Bobkov and Koldobsky [41, p. 46] and arose from the pursuit of a central limit theorem for convex bodies, as discussed briefly in Note 9.9 and finally achieved by Klartag [133]. Concerning the latter, further information is given in [16, pp. 363–5] and [50, Chapter 12], and Fresen [87] presents a short proof of Klartag's theorem, as well as a proof that it is implied by THIN.

It is known that

$$\mathsf{KLS} \Leftrightarrow \mathsf{POI} \Leftrightarrow \mathsf{EXP} \Leftrightarrow \mathsf{FMOM} \Rightarrow \mathsf{THIN} \Rightarrow \mathsf{HYP},$$

and moreover that these relations hold when μ is fixed.

To the three conjectures equivalent to KLS should be added another, due to Jiang, Lee, and Vempala [123]. This is slightly too technical to state here, but is, roughly speaking, a conjectured central limit theorem to the effect that for independent random vectors X, Y with log-concave probability densities, the random variable $X \cdot Y$ is close to a Gaussian.

That $\mathsf{KLS} \Rightarrow \mathsf{HYP}$ had apparently been observed by Ball by 2006, the argument appearing much later in joint work with Nguyen [25, Section 5]. In [25, Theorem 5.1] (see also [50, Theorem 16.1.1], which we follow here), it is shown that if X, Y are independent random variables whose probability density function is the density of an isotropic log-concave measure μ, and

$$h\left(\frac{X+Y}{\sqrt{2}}\right) - h(X) \geq \kappa\,(h(Z) - h(X))\,, \tag{22}$$

where $0 < \kappa < 1$, h denotes entropy defined by (5), and Z is the standard Gaussian random vector, then the isotropic constant $L_\mu \leq e^{1+2/\kappa}$. The desired implication now follows from $\mathsf{KLS} \Rightarrow \mathsf{POI}$ and [25, Theorem 1.1], the latter proving that if an isotropic log-concave measure μ with density f satisfies (18), then (22) holds with $\kappa = C/8$.

Denote by $\lambda_1(\mu)$ the best C such that (18) holds; see [50, Theorem 14.1.5] and the discussion preceding it. (The notation $C_P(\mu) = 1/\lambda_1(\mu)$ is sometimes used.) When μ is the normalized restriction of volume λ_n to a convex body K in $\mathbb{E}^n$, $\lambda_1(\mu) = \lambda_1(K)$, the first eigenvalue of the Laplacian on K. It is known that

$$\frac{1}{4}\mathrm{Is}_\mu^2 \leq \lambda_1(\mu) \leq \pi\,\mathrm{Is}_\mu^2. \tag{23}$$

These estimates are given by Ledoux [163, (5.8), p. 238] (see also [50, Theorems 14.1.6 and 14.1.7]), with 36 instead of π on the right-hand side. As noted by Klartag [138, p. 3], the improved constant in (23) was found by De Ponti and Mondino [69]. The inequality on the left, a form of Cheeger's inequality $\lambda_1 \geq h^2/4$ from [60] (proved independently by Maz'ya [177]), yields $\mathsf{KLS} \Rightarrow \mathsf{POI}$. That on the right, which originates in work of Buser [53], shows that $\mathsf{POI} \Rightarrow \mathsf{KLS}$. The implication $\mathsf{POI} \Rightarrow \mathsf{EXP}$ was proved by Gromov and Milman [99]. That $\mathsf{EXP} \Leftrightarrow \mathsf{FMOM} \Rightarrow \mathsf{KLS}$ is due to Milman [184], who also shows that the best constants C in (19) and (20) equal Is_μ, up to a universal constant; see also [9, Sections 1.4 and 1.5] and [50, Section 14.2], as well as [185] for more general versions of $\mathsf{FMOM} \Rightarrow \mathsf{KLS}$. To show that $\mathsf{POI} \Rightarrow \mathsf{THIN}$, take $\varphi(x) = \|x\|^2 - n$ in (18), using (2) to check that $\int_{\mathbb{E}^n} \varphi\, d\mu = 0$, and then

apply the inequality at the beginning of the proofs of [50, Lemma 12.3.1] or [75, Lemma 1.4] to conclude that

$$\sigma_n \le 2\lambda_1(\mu)^{-1/2}. \tag{24}$$

Finally, $\mathsf{THIN} \Rightarrow \mathsf{HYP}$ was demonstrated by Eldan and Klartag [75] (see also [9, Section 3.1] and [50, Theorem 12.5.1]), who prove that

$$L_\mu \le C\sigma_n \tag{25}$$

for any isotropic log-concave measure μ in $\mathbb{E}^n$. These are the only known relationships between these conjectures. However, Eldan [73] (see also [50, Section 14.6]) proved that

$$\mathrm{Is}_n \ge C\left((\log n)\sum_{k=1}^{n}\frac{\sigma_k^2}{k}\right)^{-1/2}, \tag{26}$$

a strong result towards reversing $\mathsf{KLS} \Rightarrow \mathsf{THIN}$.

We mentioned above that it suffices to prove KLS or any of its equivalent statements when μ is the normalized restriction of volume λ_n to a convex body K in $\mathbb{E}^n$. Kolesnikov and Milman [158, Theorem 12] show that in POI one can further assume that φ is a harmonic function on K. Unlike HYP, for which several reductions have been established (see, for example, [50, Section 6.2]), little else seems to be known regarding reductions of KLS.

Next, we summarize the evidence for these conjectures. Lee and Vempala [164], who also provide references for earlier bounds obtained by several authors (see also [50, Chapter 13] and [101, pp. 1044–5]), use a variant of the stochastic localization method introduced by Eldan (already mentioned in the discussion above about the proof of HYP) in proving (26). Their result is $\sigma_n \le Cn^{1/4}$, which together with (26) gives $\mathrm{Is}_n \ge C\, n^{-1/4}(\log n)^{-1/2}$. However, in a remarkable development in which he builds on the methods of [73] and [164], Chen [61] proves that $\mathrm{Is}_n \ge C\, n^{-o(1)}$. In fact, Chen actually first shows that $\sigma_n \le C\, n^{o(1)}$, the lower bound for Is_n following from (26) at the cost of a factor of $\log n$. More details about stochastic localization and the results discussed so far may be found in Eldan's survey article [74, Section 4]. By adding other techniques to Chen's method, substantial improvements on his bounds were made by Klartag and Lehec [142] (see also Jambulapati, Lee, and Vempala [121]). Like Chen, these latter authors obtained an upper bound for σ_n first. Klartag [138] again uses Chen's method, but instead applies a refined version of an inequality due to Lichnerowicz (bounding $\lambda_1(\mu)$ from below for so-called t-uniformly log concave measures) to estimate Is_n directly. His bound

$$\mathrm{Is}_n \ge C/\sqrt{\log n}, \tag{27}$$

the best currently known, yields $\sigma_n \le C\sqrt{\log n}$ (via (24) and the left inequality in (23)). A dramatic improvement over this bound for σ_n comes from the work of Guan [100], in which he proves that

$$\sigma_n \le C \log \psi_n. \tag{28}$$

(Recall that $\psi_n = 1/\mathrm{Is}_n$.) From (27) and (28), it follows that $\sigma_n \le C \log\log n$, which is currently the best known bound. Combining this with (25), we obtain $L_n \le C \log\log n$, where

$$L_n = \sup\{L_K : K \text{ is a convex body in } \mathbb{E}^n\}.$$

As was discussed earlier, Klartag and Lehec [143] used Guan's work in proving that in fact $L_n \le C$.

A first glimpse at stochastic localization might begin with the "localization lemma" of Lovász and Simonovits [169, Lemma 2.5], which reduces multidimensional integral inequalities to those in a single variable. (A similar method was used earlier by Payne and Weinberger [203].) In particular, inequalities involving integrals over convex bodies K in $\mathbb{E}^n$ are reduced, by repeated bisection (by volume) of K by hyperplanes, to integrals over line segments. While effective in some situations, this method does not preserve the isotropic property, and it is difficult to control the covariance that is then a necessary ingredient in KLS. (The conjecture for general log-concave measures is that Is_μ is bounded below by C divided by the square root of the spectral norm of the covariance of μ.) Stochastic localization begins with a log-concave measure $\mu = \mu_0$ on $\mathbb{E}^n$ instead of a convex body. Roughly speaking, instead of bisecting (by measure), μ is multiplied by a small linear function with gradient in a random direction, and this is repeated continuously to produce measures μ_t, $t > 0$, that remain log-concave and whose covariance can be controlled. A lecture by Chen [62] provides an excellent introduction illustrated by examples.

Kolesnikov and Milman [159, p. 3581] provide a list of references to works that establish KLS for various classes of convex bodies (i.e., for the normalized restriction of volume λ_n to members of these classes), to which they add generalized Orlicz balls (see also [30]). Due to the relations between the constants in KLS, POI, EXP, FMOM, THIN, and HYP described above, proving any of the first four conjectures true for a class of convex bodies implies the same for the other conjectures, and HYP must be true for any class for which THIN is true. Interestingly, it is not known whether KLS is true for unconditional convex bodies (see [50, Section 14.5]), but Klartag [134] (see also [9, Section 2.4] and [50, Proposition 12.4.1]) proves THIN for this class. Alonso-Gutiérrez and Bastero [11] verify THIN for hyperplane projections and Steiner symmetrizations of unit balls in l_p^n, $p \ge 1$, extending earlier work described in [50, Section 12.3]. Several other positive partial results are listed in [11, p. 881]; more recently, Radke and Vritsiou [208], Vritsiou [235], and Dadoun, Fradelizi, Guédon, and Zitt [65] have confirmed THIN for the unit balls of classical spaces of matrices with the operator norm, including the Schatten classes.

Alonso-Gutiérrez, Prochno, and Thäle [13, Theorem A] show that a certain statement (too technical to reproduce here) about moderate or large deviations for isotropic log-concave random vectors implies that KLS is false.

The following statement was posed as a question by Lee and Vempala [165, p. 29].

(LSOB). There is a universal constant $C > 0$ such that for every isotropic log-concave measure μ in $\mathbb{E}^n$ and every smooth function φ such that $\int_{\mathbb{E}^n} \varphi^2 \, d\mu = 1$, one has the log-Sobolev

inequality

$$C \operatorname{Ent}_\mu(\varphi^2) \le \int_{\mathbb{E}^n} \|\nabla \varphi\|^2 \, d\mu, \tag{29}$$

where

$$\operatorname{Ent}_\mu(\varphi) = \int_{\mathbb{E}^n} \varphi \log \varphi \, d\mu - \left(\int_{\mathbb{E}^n} \varphi \, d\mu \right) \log \left(\int_{\mathbb{E}^n} \varphi \, d\mu \right).$$

Setting $\varphi = 1 + \varepsilon g$ in (29) (cf. [214, p. 119]), it is not hard to check that

$$\operatorname{Ent}_\mu(\varphi^2) = 2\varepsilon^2 \operatorname{Var}_\mu(g) + O(\varepsilon^3),$$

where

$$\operatorname{Var}_\mu(g) = \int_{\mathbb{E}^n} g^2 \, d\mu - \left(\int_{\mathbb{E}^n} g \, d\mu \right)^2,$$

and $\int_{\mathbb{E}^n} \|\nabla \varphi\|^2 \, d\mu = \varepsilon^2 \int_{\mathbb{E}^n} \|\nabla g\|^2 \, d\mu$. Letting $\varepsilon \to 0$, we see that the log-Sobolev inequality (29) implies Poincaré's inequality (18) (in a slightly more general form), so via POI, we obtain

$$\mathsf{LSOB} \Rightarrow \mathsf{KLS}.$$

Moreover the constant in (18) is the same as that in (29). Stavrakakis and Valettas [229] prove that LSOB holds for the class of l_p^n balls with $2 \le p \le \infty$. However, according to E. Milman (private communication), LSOB is actually false in general, failing for example for the exponential measure with density $\|x\|_1/2^n$ and for normalized restriction of volume λ_n to B_1^n. See the article [166] by Lee and Vempala for related results.

There are at least three other conjectures that have been claimed to be at least as strong as KLS. To state the first, recall that if f and g are measurable functions on $\mathbb{E}^n$, their *infimal convolution* is defined by

$$f \,\square\, g(x) = \inf_{y \in \mathbb{E}^n} \{f(x-y) + g(y)\}.$$

The following conjecture from [162, p. 148] is sometimes called the Latała–Wojtaszczyk conjecture.

Infimal convolution conjecture (INFC). There is a universal constant $C > 0$ such that if μ is an even log-concave measure in $\mathbb{E}^n$ and $\varphi : \mathbb{E}^n \to [0, \infty]$ is measurable, then

$$\left(\int_{\mathbb{E}^n} e^{\varphi \,\square\, \left(\Lambda^* \mu\left(\frac{\cdot}{C}\right)\right)} \, d\mu \right) \left(\int_{\mathbb{E}^n} e^{-\varphi} \, d\mu \right) \le 1,$$

where

$$\Lambda^* \mu(x) = \sup_{y \in \mathbb{E}^n} \left\{ x \cdot y - \log \int_{\mathbb{E}^n} e^{y \cdot z} \, d\mu(z) \right\}, \tag{30}$$

for $x \in \mathbb{E}^n$.

The function $\Lambda^* \mu$ is sometimes called the *Cramer transform* of μ. It is the Legendre transform (see [227, p. 40]) of the *logarithmic Laplace transform* $\Lambda\mu$ of μ (the second term on the right of (30)). The function $\Lambda\mu$ is always convex with $\Lambda\mu(o) = 0$, and if μ is log-concave, it is also C^∞-smooth and strictly convex in the open set where it is finite; see [146, p. 16]. In

particular, $\Lambda^*\mu$ is a convex function. INFC is the focus of [50, Chapter 15], where it is noted that the conjecture is true for product measures, rotationally invariant measures, and uniform distributions on unit balls in l_p^n, $p \geq 1$. It is claimed in [229, p. 363] that INFC implies KLS, and this is repeated in [50, Theorem 15.3.13]. However, the proof of the latter requires that the measure μ is even, so one can only conclude that INFC implies a symmetric version of KLS in which the measure μ in (16) is even.

In fact, the basic question of whether KLS (or its equivalent version, see (17)) is equivalent to its symmetric version, in which the measure μ in (16) is even (or the convex body K in (17) is centered, respectively), seems to be open at the present time. It is known that THIN is equivalent to its symmetric version; see [9, p. 46]. As observed by E. Milman, it follows from this and (26) that one can get from the symmetric version of KLS to KLS itself at the cost of a $\log n$ factor.

If $1 \leq m \leq n$, the *lower Minkowski m-content* of a Borel set A in $\mathbb{E}^n$ is defined by

$$M_*^m(A) = \liminf_{\varepsilon \to 0+} \frac{\lambda_n(A + \varepsilon B^n)}{\kappa_{n-m}\, \varepsilon^{n-m}}.$$

See, for example, [77, p. 273].

Waist conjecture (WAI). There exists a universal constant $C > 0$ such that if $1 \leq k \leq n$ and $K \in \mathcal{K}_0^n$ with $\lambda_n(K) = 1$, there is a $\phi \in SL_n$ such that if $f : \phi K \to \mathbb{E}^k$ is continuous, there is an $x \in \mathbb{E}^k$ for which

$$M_*^{n-k}(f^{-1}(x)) \geq C^k. \tag{31}$$

WAI is stated as a question by Klartag [135, p. 131]. He proves that (31) is true with C^k replaced by C^{n-k} on the right-hand side and states without proof that

$$\text{WAI} \Rightarrow \text{KLS}.$$

Two hyperplane conjecture (2HYP). There exists a universal constant $C > 0$ such that if $K \in \mathcal{K}_0^n$ is centered and the open set $E \subset K$ with $\lambda_n(E) = \lambda_n(K)$ attains the infimum in (13), there is a half-space Γ such that

$$(\operatorname{int} K) \cap \Gamma \subset E, \ (\operatorname{int} K) \cap (-\Gamma) \subset (\operatorname{int} K) \setminus E, \quad \text{and} \quad \lambda_n(K \cap \Gamma) \geq C\, \lambda_n(K).$$

2HYP was proposed by Jerison [122, Conjecture 6.4]. Loosely speaking, it says that the interface between the isoperimetric set E for $\operatorname{int} K$ and the complement of E in $\operatorname{int} K$ is confined to the relatively small region between the hyperplanes that bound Γ and its reflection $-\Gamma$ in the origin. From [122, Proposition 6.6], due to E. Milman, one can conclude that 2HYP implies the symmetric version of the alternate form (17) of KLS. The reader of [122, p. 729] should be aware that in [122], the latter is taken to be KLS itself, but, as was mentioned above, it is not known whether KLS is equivalent to its symmetric version.

Problem 8.5

The problem remains open, but the following small observation may be relevant. Suppose L is a generalized intersection body. Then $\rho_L = R\,\mu \in C(S^{n-1})$, where μ is a signed finite even

Borel measure in S^{n-1}. Following the approach in the proof of Theorem 4.1.16, we can apply the Jordan decomposition theorem to conclude that there are finite even Borel measures $\overline{\mu}_j$ in S^{n-1}, $j = 1, 2$, such that $\mu = \overline{\mu}_1 - \overline{\mu}_2$. If

$$R\overline{\mu}_j \in C(S^{n-1}), \tag{32}$$

and $\rho_{L_j} = R\overline{\mu}_j$, for $j = 1, 2$, then L_1 and L_2 are intersection bodies such that $L_1 = L\widetilde{+}L_2$. Thus this problem has an affirmative answer whenever (32) holds.

Problem 8.6

Until recently, there was one result, pertaining to the case when $i = n - 1$. (The statement in Note 8.6 concerning the case $i = 1$ is erroneous.) Fish, Nazarov, Ryabogin, and Zvavitch [78] prove that if L is a star body in $\mathbb{E}^n$ sufficiently close to B^n in the Banach–Mazur metric, then $I^m L \to B^n$ as $m \to \infty$ in the Banach–Mazur metric. For such L, if $I^m L = cL$ for some $c > 0$ and $m \in \mathbb{N}$, then L is a centered ellipsoid.

However, when $i = n - 1$, a slight modification of the problem has now been completely solved by Milman, Shabelman, and Yehudayoff [186]. If $n \geq 3$ and the hypothesis that $I^2 L$ is homothetic to L is replaced by $I^2 L = cL$ for some $c > 0$ (a stronger, but more natural assumption), then [186, Theorem 1.1] states that L must be a centered ellipsoid. Moreover, the same result holds, modulo sets of measure zero, if L is a Borel set star-shaped with respect to the origin; see [186, Remark 1.4]. The proofs depend on a novel use of continuous Steiner symmetrization, first employed by Rogers [212] in proving the result often called the Brascamp-Lieb-Luttinger inequality. The authors point out that if $n = 2$, then $I^2 L = 4L$ holds for any centered star body.

Problem 8.7

Regarding the case $i = 1$, the statement in Note 8.6 is misleading. Grinberg and Zhang [337book, Corollary 9.8] show that when $n \geq 3$ (the condition is omitted), if $I_1 L = cL$ for some $c > 0$ (note that this implies that L is centered), then L is a centered ball. However, this nice result has no bearing on Problem 8.6.

With the slightly stronger assumption that $IL = cL$ for some $c > 0$, Milman, Shabelman, and Yehudayoff [186, Corollary 1.2] provide a complete solution: If $n \geq 3$, then L must be a centered ball. As with their solution of Problem 8.6, the same result holds, modulo sets of measure zero, if L is a Borel set star-shaped with respect to the origin; see [186, Remark 1.4]. When $n = 2$, we have $IL = 2L$ whenever L is invariant under a rotation by $\pi/2$ about the origin. The authors show that their result for $n \geq 3$ is a consequence of their solution of Problem 8.6. Indeed, if $IL = cL$ for some $c > 0$, then $I^2 L = I(cL) = c^{n-1} IL = c^n L$, so L is a centered ellipsoid by [186, Theorem 1.1]. Then $L = \phi B^n$ for some $\phi \in GL_n$, so by Theorem 8.1.6, we have

$$c\,\phi B^n = cL = IL = |\det \phi| \phi^{-t} IB^n = |\det \phi| \kappa_{n-1} \phi^{-t} B^n.$$

Therefore $\phi^t \phi B^n = (\kappa_{n-1}/c)|\det \phi| B^n$, so up to scaling, ϕ is an orthogonal transformation and hence L is a centered ball.

Problem 8.8

Gardner, Ryabogin, Yaskin, and Zvavitch [90] answer part (i) of this problem by constructing coaxial convex bodies of revolution in $\mathbb{E}^n$, $n \geq 3$, such that one body is centered and the other is not centrally symmetric, while both have the same inner section function. Shortly after, Nazarov, Ryabogin, and Zvavitch [196] went further by constructing an asymmetric convex body of revolution in $\mathbb{E}^n$, $n \geq 3$, with a constant inner section function, thereby solving part (ii) of the problem. (For $n = 4$, this was achieved by the same authors earlier in [195].)

Problem 8.9

Nazarov, Ryabogin, and Zvavitch [195] provide a strong negative answer to part (i) of this problem when $n \geq 4$ is even, by constructing non-congruent coaxial bodies of revolution K and L with $\Pi K = \Pi L$, $CK = CL$, and $IK = IL$. Parts (ii) and (iii) appear to be open.

Problem 9.1

The problem is also stated by Schneider [227, (10.80), p. 570], who provides further commentary in [227, Notes 2 and 6, pp. 576–8].

Petty's conjectured projection inequality remains open and is one of the great challenges in convex geometry. In view of the Blaschke–Santaló inequality for centered convex bodies, Theorem 9.2.11, it is stronger than the Petty projection inequality, Theorem 9.2.9. The fact that the extremal bodies must be zonoids K such that $\Pi^2 K$ is homothetic to K, already mentioned in Note 9.4, is detailed in [227, pp. 570–1]. This connection allows Saroglou and Zvavitch [226, Theorem 1.3] to prove that in a certain technical sense (see the report for Problem 4.4 above and the remarks in [226, p. 616]), ellipsoids are locally extremal bodies.

Saroglou [223, Theorem 3] proves that the usual Steiner symmetrization technique cannot be applied, at least in the obvious way, by showing that for $n \geq 3$, there exist $K \in \mathcal{K}_0^n$ and $u \in S^{n-1}$ such that $\lambda_n(\Pi(S_u K)) > \lambda_n(\Pi K)$. He also proves that Petty's conjectured projection inequality holds in $\mathbb{E}^3$ when K is a cone or a centered double cone.

Problem 9.2

Mahler's conjecture, still an outstanding open problem in convex geometry, has received a great deal of attention in recent years. Here we shall only update and add some details to the report of Schneider [227, pp. 564–6], who summarizes developments prior to 2014.

Mahler's conjecture has two flavors, one (MAH1) for convex bodies containing the origin in their interiors (not just those with centroid at the origin, as stated in Problem 9.2) and the other (MAH2) for centered bodies. As Schneider points out, Mahler [173] actually only published MAH2, while MAH1 appeared in print much later, in [5, p. 150] and [102, (10.9), p. 59]. Some are of the opinion that MAH1 may be easier, because $\triangle_n$, a regular simplex in $\mathbb{E}^n$ with unit volume and centroid at the origin, is thought likely to be the unique extremal body, up to a nonsingular linear transformation, while for MAH2, the extremal bodies are conjectured to be the Hanner polytopes (the term deriving from Hanner's article [107], but also sometimes associated to the names Hansen and Lima [108]), up to nonsingular linear transformations. We follow Schneider [227, pp. 564–5] in defining Hanner polytopes recursively: Closed line segments are Hanner polytopes, and a centered polytope in $\mathbb{E}^n$ is a Hanner polytope if it is

either the convex hull of the union, or the sum, of two centered Hanner polytopes contained in complementary subspaces. In $\mathbb{E}^2$ and $\mathbb{E}^3$, the only Hanner polytopes are centered cubes and octahedra, but there are many more in higher dimensions. To the early references provided in Note 9.4, it should be added that Meyer [182] also confirmed the anticipated equality case when $n = 2$.

The quantity $v(K) = \lambda_n(K)\lambda_n(K^*)$ is often called the *volume product* or *Mahler volume* of K. The Bourgain–Milman bound mentioned in Note 9.4 is that $v(K) \geq c_1^n\, v(C_n)$ when K is centered, where $C_n = [-1/2, 1/2]^n$ and c_1 is a universal constant, but Bourgain and Milman also showed that $v(K) \geq c_2^n v(\triangle_n)$ for arbitrary K containing the origin in its interior, where c_2 is a universal constant. Proofs are also given in [16, Theorems 8.2.2 and 8.5.1] and [50, Theorem 7.4.3]. See also the recent survey paper on the volume product by Fradelizi, Meyer, and Zvavitch [83], which overlaps significantly with the report below on the current status of Mahler's conjecture; moreover, it covers upper as well as lower bounds for the volume product and gives some details of proofs of some of the main results.

Mahler's conjecture received wide exposure from a 2007 post by Tao [230] in his blog. This article and the thread of comments it generated are still entertaining and informative. Even known results are continually being revisited. For example, both Meckes [181] and Saroglou [224] give different proofs of Reisner's result for zonoids mentioned in Note 9.4, Meckes via Holmes–Thompson intrinsic volumes (see Note 4.1 for a special case) and Saroglou employing shadow systems. A probabilistic proof of Mahler's conjecture for planar centered bodies is given by Tointon [232] and Rebello Bueno [209] finds a stochastic inequality in the plane that implies MAH2 when $n = 2$.

Proving MAH1 (or MAH2) up to a sub-exponential factor, i.e., showing that for all $n \geq 2$, $v(K) \geq c_n\, v(\triangle_n)$ (or $v(K) \geq c_n\, v(C_n)$, respectively) where $c_n^{1/n} \to 1$ as $n \to \infty$, is enough to establish MAH1 (or MAH2, respectively) itself. This observation is due to Tao [231, p. 219], who uses the functional analogues just mentioned together with what he calls the "tensor power trick." The same trick underlies a proof of this fact for MAH2 given independently by Berezovik and Karasev [31, Theorem 2.1].

In a remarkable work, Iriyeh and Shibata [114] prove MAH2 in $\mathbb{E}^3$, both the inequality and the fact that equality holds if and only if either K or K^* is a centered parallelepiped. Aspects of the Iriyeh–Shibata theorem are simplified by Fradelizi, Hubard, Meyer, Roldán-Pensado, and Zvavitch [80], who also provide a stability version. Apart from this, only the following partial results, which nevertheless represent a very significant effort, are known.

Kuperberg [161] proved the so-called bottleneck conjecture, which asserts that the volume of a certain domain $K^\diamondsuit \subset K \times K^* \subset \mathbb{E}^{2n}$, where $K \in \mathcal{K}_0^n$ is centered, is minimized if and only if K is an ellipsoid. (In its original form posed by Kuperberg [484book], $K^\diamondsuit = \operatorname{conv}(K^+ \cup K^-)$, where $K^\pm = \{(x, y) \in K \times K^* : x \cdot y = \pm 1\}$, while in [161], the convex hull is replaced by the union of line segments joining points in K^+ to points in K^-.) This leads to improvements in the Bourgain–Milman bounds. In [161, Corollaries 1.6 and 1.8], Kuperberg shows that one can take $c_1 = \pi/4$ in the centered case and $c_2 = \pi/(2e)$ in the general case. These are currently the best bounds known when $n \geq 4$. Nazarov [193] found a completely different approach

based on multivariable complex and Fourier analysis, which yields the slightly weaker constant $c_1 = (\pi/4)^3$ for the centered case. Variations of Kuperberg's and Nazarov's proofs, and commentary of various lengths, are provided by Berndtsson [32], [33], Błocki [37, Section 9], [38], [39], Fradelizi, Meyer, and Zvavitch [83], Mastrantonis and Rubinstein [176] (who also extend Nazarov's method to the general case), and Ryabogin and Zvavitch [221]. (We note in passing that Bianchi and Kelly [35] find a proof of the Blaschke–Santaló inequality for centered convex bodies, Theorem 9.2.11, based on Nazarov's approach.) Yet another way of recovering the Bourgain–Milman bounds, using techniques generally more familiar in convex geometry, is due to Giannopoulos, Paouris, and Vritsiou [97].

Nazarov, Petrov, Ryabogin, and Zvavitch [194] prove that C_n is a strict local minimizer for the volume product in the class of centered convex bodies endowed with the Banach–Mazur metric. Kim and Reisner [130] show that $\triangle_n$ is a strict local minimizer in the class of all convex bodies, and Kim [129] significantly extends the result in [194] by showing that it holds not just for C_n but for all Hanner polytopes. Further evidence for the conjectured minimizers is obtained by Reisner, Schütt, and Werner [211], who prove that such a body cannot be C^2_+ at any point of its boundary. Related results were obtained by Harrell, Henrot, and Lamboley [109]. Also related are the quantitative lower bounds for the volume product in terms of curvature obtained by Nakamura and Tsuji [192] via a connection to Ornstein–Uhlenbeck flow.

Mahler's conjecture has been verified for some special classes of bodies. Summaries are given by Alexander, Fradelizi, García-Lirola, and Zvavitch [6], who study a related problem in finite metric spaces, and by Karasev [126], who employs symplectic methods to deal with the case of central hyperplane sections or projections of l_p balls or of Hanner polytopes. In addition to these and zonoids (see Note 9.4), the list includes bodies of revolution, polytopes (or centered polytopes) in $\mathbb{E}^n$ with at most $n+3$ vertices (or at most $2n+2$ vertices, respectively), and unconditional bodies and some generalizations of them; see [20] and [83, Section 3.6] for references. Barthe and Fradelizi [29] confirm the conjecture for bodies invariant under the action of a Coxeter group. This line of investigation was continued by Iriyeh and Shibata [115], who study the variant of Mahler's conjecture in which the minimum of $v(K)$ is sought over those $K \in \mathcal{K}^n_0$ that are invariant under a discrete subgroup G of $O(n)$. Note that MAH1 and MAH2 correspond to $G = \{\mathrm{Id}\}$ and $G = \{\mathrm{Id}, -\mathrm{Id}\}$, respectively, where Id is the $n \times n$ identity matrix. They focus on the case $n = 3$ and present the state of play in a table in [115, p. 5]. In [116], the same authors deal with the case when G is the special orthogonal group of the cube or simplex in $\mathbb{E}^n$. They obtain further results in the same vein in [117].

Functional versions of the inequalities at hand go back to Ball's PhD thesis, where in [23, Corollary 4.5] he proved a functional version of Theorem 9.2.11, the Blaschke-Santaló inequality for centered convex bodies. Corresponding versions of the Bourgain-Milman theorem were established by Artstein-Avidan, Klartag, and Milman [18] and Fradelizi and Meyer [82], and in the latter paper the following functional analogues of MAH1 and MAH2 are proposed.

If $\varphi : \mathbb{E}^n \to \mathbb{R} \cup \{\infty\}$ is a convex function such that $0 < \int_{\mathbb{E}^n} e^{-\varphi(x)}\,dx < \infty$, then

$$\int_{\mathbb{E}^n} e^{-\varphi(x)}\,dx \int_{\mathbb{E}^n} e^{-\mathcal{L}\varphi(x)}\,dx \geq e^n,$$

and for even φ, e^n can be replaced by 4^n.

Here $\mathcal{L}\varphi$ is the Legendre transform of φ, defined for $x \in \mathbb{E}^n$ by $\mathcal{L}\varphi(x) = \sup_{y\in\mathbb{E}^n}(x\cdot y - \varphi(y))$. These conjectures are shown to be equivalent to MAH1 and MAH2 in [82, Section 2], and precise equality conditions are proposed that correspond to those conjectured for MAH1 and MAH2. Fradelizi and Nahkle [84] prove the conjectured inequality and equality condition for even functions when $n = 2$ and generalize this result to certain even s-concave functions, $s > -1/n$, in [85]. The survey [83, Sections 4 and 5] (see also [227, pp. 523–4]) discusses various aspects of the functional inequalities in some detail, and corresponding inequalities involving more than one body or function are treated in [83, Section 6]. The Mahler conjecture for various classes of functions and in other settings are examined by Fradelizi, Gozlan, Sadovsky, and Zugmeyer [79].

In the remainder of this report, we shall discuss various other results and conjectures related to MAH1 or MAH2.

Chan and Pak [57] note that an inequality for finite posets called the reverse Sidorenko inequality is a special case of MAH1. See also the article by Artstein-Avidan, Sadovsky, and Sanyal [20].

Some L^p versions are formulated by Berndtsson, Mastrantonis, and Rubinstein [34], who introduce ideas of interest beyond Mahler's conjecture. If K is a compact (not necessarily convex) body in $\mathbb{E}^n$ and $p > 0$, they define its *L^p-support function* $h_{p,K}$ by

$$h_{p,K}(x) = \log\left(\frac{1}{\lambda_n(K)}\int_K e^{p\,x\cdot y}\,dy\right)^{1/p}$$

for $x \in \mathbb{E}^n$. In [34, Section 2], it is shown that $h_{p,K}$ is a convex function (even if K is not convex), that $h_{p,K} \leq h_{q,K} \leq h_K$ for $p \leq q$, and that $\lim_{p\to\infty} h_{p,K} = h_K$. In [34, Section 3], the *L^p-polar body* $K^{*,p}$ of K is defined via the gauge function

$$\|x\|_{K^{*,p}} = \left(\frac{1}{(n-1)!}\int_0^\infty e^{-h_{p,K}(rx)} r^{n-1}\,dr\right)^{-1/n}$$

for $x \in \mathbb{E}^n$, so that $K^{*,p} = \{x \in \mathbb{E}^n : \|x\|_{K^{*,p}} \leq 1\}$. Using the fact that $\int_0^\infty e^{-as}s^{n-1}\,ds = (n-1)!\,a^{-n}$, one sees that $\lim_{p\to\infty}\|x\|_{K^{*,p}} = h_K(x) = \|x\|_{K^*}$ and hence $\lim_{p\to\infty} K^{*,p} = K^*$. One also has $K^* \subset K^{*,q} \subset K^{*,p}$ when $p \leq q$. Unlike the usual polar operation, ^{*,p} is not a duality, since $(K^{*,p})^{*,p} \neq K$, in general, even when K is a centered convex body. With these concepts in hand, the authors define the *L^p-Mahler volume* $\mathcal{M}_p(K)$ of K by

$$\mathcal{M}_p(K) = n!\,\lambda_n(K)\lambda_n(K^{*,p}) = \lambda_n(K)\int_{\mathbb{E}^n} e^{h_{p,K}(x)}\,dx.$$

Note that as $p \to \infty$, $\mathcal{M}_p(K)$ approaches $n!\,v(K)$ from above; the authors justify the $n!$ factor in their definition by noting that $\mathcal{M}_p(K \times L) = \mathcal{M}_p(K)\mathcal{M}_p(L)$. By [34, Lemma 4.6],

$\mathcal{M}_p(\phi K) = \mathcal{M}_p(K)$ for any $\phi \in GL_n$. Theorem 9.2.11, the Blaschke-Santaló inequality for centered convex bodies, is generalized in [34, Theorem 1.6], which states that $\mathcal{M}_p(K) \leq \mathcal{M}_p(B^n)$ for $0 < p \leq \infty$, but for finite p the precise equality condition remains open. In [34, Conjectures 1.3 and 1.4], the authors conjecture that when $0 < p \leq \infty$,

$$\mathcal{M}_p(K) \geq \mathcal{M}_p(C_n) \tag{33}$$

for centered convex bodies K in $\mathbb{E}^n$, and

$$\mathcal{M}_p(K) \geq \mathcal{M}_p(\triangle_n) \tag{34}$$

for arbitrary convex bodies K in $\mathbb{E}^n$. (The case $p = 1$ of (33) was first conjectured by Błocki [38, p. 56] and (34) appeared in a different form in [176, Conjecture 10].) They further conjecture that the lower bounds in (33) and (34) are attained if and only if K is an image of C_n (or $\triangle_n$, respectively) under a nonsingular linear transformation. In [34, Lemma 3.12], it is shown that (34) for all $p \in (0, \infty)$ implies MAH1 and (33) for all $p \in (0, \infty)$ implies MAH2.

The Mahler volume turns up in the following conjecture that arose from discussions between R. Freij, M. Henze, G. Kalai, M. Schmidt, and G. Zieger; see [76, Conjecture 10.1].

If P is a centrally symmetric n-dimensional convex polytope in $\mathbb{E}^n$, then

$$|\mathrm{Flags}(P)| \geq \frac{(n!)^2}{2^n} v(P). \tag{35}$$

Here $\mathrm{Flags}(P)$ denotes the set of flags of P, i.e., sequences $(F_0, F_1, \ldots, F_n)$ of faces of P such that $\dim F_i = i$ and $F_i \subset F_{i+1}$ for $i = 0, 1, \ldots n-1$. Since MAH2 says that $v(P) \geq 4^n/n!$, it and (35) would prove Kalai's *flag number conjecture*, which reads as follows.

If P is a centrally symmetric n-dimensional convex polytope in $\mathbb{E}^n$, then

$$|\mathrm{Flags}(P)| \geq 2^n n!. \tag{36}$$

This is a special case of [124, Conjecture C]; the general conjecture, as mentioned in [76, Introduction], is known to be false. Equality holds in (35) and (36) if and only if P is a Hanner polytope. The relationship between these conjectures is strengthened by a further one due to Faifman, Vernicos, and Walsh [76, Conjecture 1.1], set in the framework of Funk geometry:

Let K be a convex body in $\mathbb{E}^n$, let $r > 0$, and let $\mathrm{Vol}_K(B_K(r))$ denote the (Holmes–Thompson) volume of the (forward) ball in the Funk geometry of radius r centered at o. Then $\mathrm{Vol}_K(B_K(r))$ is minimized when K is an n-dimensional simplex with centroid at o. If K is centrally symmetric, $\mathrm{Vol}_K(B_K(r))$ is minimized when K is a Hanner polytope.

A Funk metric defined on the interior of a convex body in $\mathbb{E}^n$ (not a true metric, but a non-reversible, i.e., not symmetric, pseudometric) gives rise to a particular type of Finsler geometry. The lack of symmetry necessitates defining both forward and backward balls. More detailed information and background may be found in [76], [83, Section 7.3], and [202], for

example. It is known that as $r \to 0$, $\mathrm{Vol}_K(B_K(r))$ is asymptotic to $r^n v(K)$, and in [76, Theorem 1.2], it is proved that if P is an n-dimensional convex polytope, then

$$\lim_{r\to\infty} \frac{\kappa_n}{r^n}\mathrm{Vol}_P(B_P(r)) = \frac{|\mathrm{Flags}(P)|}{n!}.$$

It follows that [76, Conjecture 1.1] would yield MAH1 and MAH2 as $r \to 0$ and (36) as $r \to \infty$. In [76, Theorem 1.3], the conjecture is proved true for unconditional convex polytopes.

The report for Problem 8.3, the slicing problem, includes the direct link found by Klartag [136], namely

$$\mathsf{SISO1} \Rightarrow \mathsf{MAH1},$$

where SISO1 denotes the sharp isotropic constant conjecture for arbitrary convex bodies. See Figure 1, which also illustrates the relations between other conjectures related to Mahler's conjecture. Fradelizi and Marín Sola [81] find a functional version of Klartag's result: If $\widehat{P}(n)$ holds, then the functional version of MAH1 stated above also holds in $\mathbb{E}^n$. Here $\widehat{P}(n)$ and a variation, $P(n)$, are the statements introduced shortly after that of SISO1 in the report for Problem 8.3. It is explained in [81] why this result does not work when $\widehat{P}(n)$ is replaced by $P(n)$.

Iriyeh and Shibata [114, Section 1.2] note that their result on MAH2 bears on Viterbo's conjectured isoperimetric inequality for the symplectic capacities of convex bodies in $\mathbb{E}^{2n}$ endowed with the standard sympletic structure. A *symplectic capacity* c is a set function with the following properties: monotonicity, invariance under symplectomorphisms, positive homogeneity of degree 2, and the normalization $c(B^{2n}) = c(B^2 \times \mathbb{E}^{2n-2}) = \pi$. The smallest symplectic capacity is the *Gromov width*

$$c_G(K) = \sup\{\pi r^2 : \ rB^{2n} \text{ embeds symplectically into } K\},$$

and the largest symplectic capacity is the *cylindrical capacity*

$$c_Z(K) = \inf\{R : \ K \text{ embeds symplectically into } Z^{2n}(R)\},$$

where $Z^{2n}(R)$ is the cylinder $\{z \in \mathbb{C}^n : \pi|z_1|^2 \le R\}$. Further examples are listed in [1] and [103], where more information and references can be found. Several non-equivalent symplectic capacities agree on convex bodies. In particular, [103, Theorem 1.12] (ascribed there to several different authors) implies that if K is a convex body in $\mathbb{E}^{2n}$, then

$$c_{HZ}(K) = c_{SH}(K) = c^1_{EH}(K), \tag{37}$$

where c_{HZ}, c_{SH}, and c^1_{EH} are the *Hofer–Zehnder capacity*, the *Viterbo capacity*, and the first *Ekeland–Hofer capacity*, respectively. Their common value is often called the *Ekeland–Hofer–Zehnder capacity* and denoted by $c_{EHZ}(K)$. Formulas for this quantity, including a combinatorial expression when K is a polytope, are provided by Haim-Kislev [105].

Viterbo's conjecture [234, p. 426] is as follows.

Viterbo's conjecture (VIT). If $K \in \mathcal{K}^{2n}$ and c is any symplectic capacity, then

$$\frac{c(K)}{c(B^{2n})} \le \left(\frac{\lambda_{2n}(K)}{\lambda_{2n}(B^{2n})}\right)^{1/n}. \tag{38}$$

We shall label the *symmetric case of Viterbo's conjecture*, i.e., the conjecture restricted to centrally symmetric $K \in \mathcal{K}^{2n}$, by VIT2.

The surprising connection with Mahler's conjecture was discovered by Artstein-Avidan, Karasev, and Ostrover [17, Theorem 1.6], who showed that

$$\mathsf{VIT2} \Rightarrow \mathsf{MAH2}.$$

The proof follows easily from $c(B^{2n}) = \pi$ and the fact, proved in [17, Theorem 1.7], that the Hofer–Zehnder capacity c_{HZ} of $K \times K^*$, where $K \in \mathcal{K}^n$ is centered, equals 4. Conversely, MAH2 implies VIT2 when $c = c_{HZ}$ for bodies of this form, so the Iriyeh–Shibata theorem yields this special case of VIT2 when $n = 3$; see [114, Corollary 1.3].

It was mentioned above that MAH1 and MAH2 are unaffected by the insertion of a subexponential factor on their right-hand sides. Berezovik and Karasev [31, Theorem 7.1] prove an analogous result for the following consequence of VIT:

$$\lambda_{2n}(K) \ge \frac{c_{EHZ}(K)^n}{n!}, \tag{39}$$

where K is a convex body in $\mathbb{E}^{2n}$ and c_{EHZ} is the Ekeland-Hofer-Zehnder capacity. The authors define the *symplectic polar body* K^ω of a convex body K in $\mathbb{E}^{2n}$ containing the origin in its interior by

$$K^\omega = \{x \in \mathbb{E}^{2n} : \omega(x, y) \le 1 \text{ for all } y \in K\},$$

where ω is the standard symplectic form on $\mathbb{E}^{2n}$. They note that $K^\omega = JK^*$, where J is the standard complex structure rotation, and conjecture that when $K^\omega = K$ (a condition they show implies that K is centrally symmetric), $\lambda_{2n}(K) \ge 2^n/n!$. This conjecture is shown in [31, Theorem 4.3] to be equivalent to MAH2. In view of (39), the implication VIT2 $\Rightarrow$ MAH2 follows if $c_{EHZ}(K) \ge 2$ whenever $K^\omega = K$, but [31, Theorem 5.1] provides an even stronger result, that in fact $c_{EHZ}(K) \ge 2 + 1/n$ whenever K is centrally symmetric and $K^\omega \subset K$.

Viterbo [234, Theorem 5.1] proved that (38) holds with a factor linear in n on the right-hand side. This was substantially improved by Artstein-Avidan, Milman, and Ostrover [19], who proved that (38) holds up to a universal constant factor. As noted in [2, p. 692], (38) holds when $c = c_G$, the Gromov width. It is also true for ellipsoids, and Abbondandolo and Benedetti [1, Corollary 2] prove that (38) holds for sufficiently smooth K in a neighborhood of B^{2n}. Cases where equality holds in (38) are discussed by Balitskiy [22] and Karasev and Sharipova [127]. Rudolf [216] finds reformulations of VIT that are variants of Moser's still-unsolved worm problem.

In [3, Section 4], Akopyan, Balitskiy, Karasev, and Sharipova show that MAH1 does not follow from VIT. However, Haim-Kislev and Ostrover [106] prove that VIT is actually false for all $n \ge 2$. Before describing their counterexample, we mention another related conjecture.

The fact (see (37)) that several non-equivalent symplectic capacities agree on convex bodies suggests that perhaps they all do, and this conjecture has been around for some time (see [110, Conjecture 1.9], [178, Problem 53, p. 572], and [199, Conjecture 5.1]). According to Hermann [110, Remark 1.8], it is probably over 30 years old. Gutt, Hutchings, and Ramos [103] call it the *strong Viterbo conjecture*. They prove, among other results, that it holds in $\mathbb{E}^4$ for a class of bodies they call monotone toric domains. Cristofero–Gardiner and Hind [64] generalize this result to higher dimensions. Edtmair [72] proves the conjecture for C^3 convex bodies in $\mathbb{E}^4$ in a neighborhood of B^4. However, the conjecture is false for all $n \geq 2$, because if it were true, the fact that Gromov width c_G satisfies (38) would imply that VIT is true. Nevertheless, the following remains a possibility.

Symmetric strong Viterbo conjecture (SSV). All symplectic capacities agree on centrally symmetric convex bodies.

Again since c_G satisfies (38), we have

$$\mathsf{SSV} \Rightarrow \mathsf{VIT2}.$$

Let P be the regular pentagon $P = \operatorname{conv}\{e^{2k\pi i/5}, k = 0, \dots, 4\} \subset \mathbb{E}^2$ and let ϕ be the clockwise rotation by $\pi/2$ about the origin in $\mathbb{E}^2$. Haim-Kislev and Ostrover [106, Proposition 1.3] find that

$$c_{EHZ}(P \times \phi P) = 2 \cos \tfrac{\pi}{10} \left(1 + \cos \tfrac{\pi}{5}\right).$$

Since $\lambda_2(P) = (5/2) \sin(2\pi/5)$, a computation yields

$$\frac{c_{EHZ}(P \times \phi P)}{2\lambda_4(P \times \phi P)} = \frac{c_{EHZ}(P \times \phi P)}{2\lambda_2(P)^2} = \frac{\sqrt{5}+3}{5} > 1,$$

contradicting (38).

It is worth mentioning that there is evidence that the utility of symplectic methods in convex geometry may extend far beyond Mahler's conjecture. In a fascinating paper, Akopian, Karasev, and Petrov [4] show that a conjectured weak finite subadditivity property of the Hofer–Zehnder capacity would recover the results of Ball on Bang's plank problem, another major open question from a different area of convex geometry.

A further development hails from the birth of systolic geometry in 1949, when Loewner discovered that

$$\frac{\operatorname{area}(\mathbb{T}^2)}{\operatorname{sys}(\mathbb{T}^2)^2} \geq \frac{\sqrt{3}}{2},$$

where $\mathbb{T}^2$ is the 2-torus equipped with any Riemannian metric g, and the *systole* sys denotes the least length of a noncontractible loop. Loewner's inequality appeared in print a few years later, in an article by his student Pu [206], who proved the corresponding inequality for the projective plane $\mathbb{P}^2$, with constant $2/\pi$ on the right-hand side and equality precisely for the round geometry. In the convex geometry setting, Pu's theorem implies that if K is a centrally symmetric convex body in $\mathbb{E}^3$ with sufficiently smooth boundary, then $\lambda_2(\partial K)/\operatorname{sys}(\partial K)^2 \geq 1/\pi$, where $\operatorname{sys}(\partial K)$ is the least length of a closed geodesic; see, for example, [14, p. 649]. Equality holds when $K = B^3$, but there are non-symmetric convex bodies K for which the

ratio $\lambda_2(\partial K)/\mathrm{sys}(\partial K)^2$ is arbitrarily close to $1/2\sqrt{3} < 1/\pi$. Inequalities such as Loewner's and Pu's are termed *isosystolic inequalities* and can be viewed as reverse forms of standard isoperimetric inequalities. Álvarez Paiva, Balacheff, and Tzanev [15, Conjecture II and p. 968] make the following conjecture.

(ABT). If a reversible optical hypersurface in the cotangent bundle of $\mathbb{P}^n$ encloses a volume V, it carries a periodic characteristic whose action is at most $V^{1/n}/2$. Moreover, this short characteristic can be chosen so that its projection onto the base manifold is a non-contractible closed geodesic.

(In [15], V is $n!$ times the usual symplectic volume.) Reversible optical hypersurfaces are those whose intersections with cotangent spaces are quadratically convex (i.e., all osculating quadratics are ellipsoids) and centered closed hypersurfaces. As is explained in [15], statements such as ABT have implications in the geometry of numbers. According to J. C. Álvarez Paiva (private communication), a stronger "folklore" conjecture may be formulated in terms of Finsler geometry:

Finsler metric periodic geodesic conjecture (FPG). A Finsler metric on n-dimensional real projective space $\mathbb{P}^n$ with the same Holmes–Thompson volume as the canonical metric carries a non-contractible periodic geodesic whose length is at most π.

By [15, p. 968], ABT is the special case of FPG that applies to reversible Finsler metrics F (those for which $F(-v) = F(v)$ for all tangent vectors v). This fact and [15, Theorem VII] means that

$$\mathsf{FPG} \Rightarrow \mathsf{ABT} \Rightarrow \mathsf{MAH2}.$$

No relation between FPG and MAH1 is known. In the same paper in which they prove the local version of VIT mentioned above, Abbondandolo and Benedetti [1] also establish a local version of FPG. For $n = 2$, ABT was proved by Ivanov [120], who thereby extended Pu's inequality. In the language of contact geometry, an odd-dimensional counterpart of symplectic geometry, both ABT and VIT can be viewed as saying that under an appropriate convexity assumption, "Zoll contact forms, i.e., those such that all the orbits of the induced Reeb flow are periodic with the same period, have minimal contact systolic volume," where the systolic volume is the general term for quantities such as those on the left-hand sides of Loewner's and Pu's inequalities. Despite this commonality, no direct implication between ABT and VIT seems to be known, nor has a natural conjecture been formulated that implies both of them.

Problem 9.3

This problem, also stated by Schneider [227, (9.57), p. 515], has now been completely solved by E. Milman and Yehudayoff [187]. Before describing their admirable results, we summarize a couple of earlier developments.

Dafnis and Paouris [67] introduced the *normalized ith affine quermassintegral*, which can be defined for $K \in \mathcal{K}_0^n$ and $1 \le i \le n-1$ by

$$\Phi_{[i]}(K) = \lambda_n(K)^{-1/n} \left(\frac{\kappa_n}{\kappa_i}\right)^{1/i} \Phi_{n-i}(K)^{1/i},$$

and noted that Problem 9.3 is equivalent to asking whether

$$\Phi_{[i]}(K) \geq \Phi_{[i]}(B^n),$$

with equality if and only if K is an ellipsoid. They asked whether there exists a universal constant $c_1 > 0$ such that

$$\Phi_{[i]}(K) \geq c_1\sqrt{n/i},$$

for $1 \leq i \leq n-1$. This was proved by Paouris and Pivovarov [201, Theorem 5.1], providing an affirmative answer to Problem 9.3 in the asymptotic sense, but this is now a consequence of [187, Theorem 1.2].

Zou and Xiong [239] prove the inequality

$$\kappa_n^{(n-i)/n}\lambda_n(P_iK)^{i/n} \leq \Phi_{n-i}(K),$$

for $1 \leq i \leq n-1$, where P_iK, $1 \leq i \leq n$, is an ellipsoid they call the *ith projection mean ellipsoid*, defined via a constrained optimization problem. Equality holds when $2 \leq i \leq n-1$ if and only if K is an ellipsoid and when $i = 1$ if and only if there is a $\phi \in SL(n)$ such that ϕK has constant width. They note that P_nK is the so-called Petty ellipsoid, i.e., the L_1 John ellipsoid (see p. 381). It is known that $\lambda_n(K) \geq \lambda_n(P_nK)$ but the authors show that it is not generally true that $\lambda_n(K) \geq \lambda_n(P_iK)$ when $1 \leq i \leq n-1$.

As was mentioned above, Milman and Yehudayoff [187] have now solved Problem 9.3. Moreover, their results are stronger in two ways. Firstly, they prove in [187, Theorems 1.2 and 5.1] that for $1 \leq i \leq n-1$ and $u \in S^{n-1}$, we have

$$\Phi_{n-i}(K) \geq \Phi_{n-i}(S_uK),$$

with equality for all $u \in S^{n-1}$ if and only if K is an ellipsoid, where S_uK is the Steiner symmetral of K in the direction u. This surprising result is new even for $i = n-1$. (Note that in [187] the affine quermassintegrals are indexed differently to the usual custom.) This yields the desired inequality via the usual method of successive Steiner symmetrizations (see Lemma 9.2.3). Secondly, they show in [187, Theorem 1.3] that among all convex bodies in $\mathbb{E}^n$ of a given volume, ellipsoids are the only local minimizers of Φ_{n-i} with respect to the Hausdorff metric.

In addition, Milman and Yehudayoff [187, Section 8.2] also find a short proof of the Petty projection inequality, Theorem 9.2.9, that avoids use of the Busemann–Petty centroid inequality, Corollary 9.2.7; instead, their proof is analogous to the one presented for Theorem 9.2.11, the Blaschke–Santaló inequality for centered convex bodies, which was found by Meyer and Pajor [605book, Lemma 1]. In particular, they prove that for $u \in S^{n-1}$ and $y \in u^\perp$, we have

$$\lambda_1((\Pi^*K) \cap (l_u + y)) \leq \lambda_1((\Pi^*S_uK) \cap (l_u + y)).$$

Interestingly, this can be expressed as $X_u(\Pi^*K) \leq X_u(\Pi^*S_uK)$, where X_u denotes the X-ray in the direction u, and similarly, (9.2) on p. 362 is equivalent to $X_{u^\perp}K^* \leq X_{u^\perp}S_u^*K$, where $X_{u^\perp}$ signifies the $(n-1)$-dimensional X-ray parallel to $u^\perp$. Yet another formulation of the two results would be $S_u(\Pi^*K) \subset S_u(\Pi^*S_uK)$ and $S_{u^\perp}K^* \subset S_{u^\perp}S_u^*K$, where $S_{u^\perp}$ denotes Schwarz symmetrization parallel to $u^\perp$.

Problem 9.4

Milman and Yehudayoff [187, Theorem 8.2] provide an affirmative answer when $i = 1$, thus establishing for the first time a sharp Blaschke–Santaló inequality for compact sets. In [187, Section 8.1], they comment on the possibility of similarly extending their results for convex bodies to compact sets when $1 < i \leq n-1$, but leave these cases open.

Problem 9.5

The conjectured inequality is also stated by Schneider [227, (9.56), p. 515]. As is pointed out in Note 9.6, the special case $i = 0$ reduces to Problem 9.3. Apart from this, the problem is still wide open.

Milman and Yehudayoff [187] introduce the *L_p-moment affine quermassintegrals*, defined (in our notation) for $K \in \mathcal{K}_0^n$, $0 \leq i \leq n-1$, and $0 > p \geq -n$, by

$$\Phi_{n-i,p}(K) = \frac{\kappa_n}{\kappa_i} \left(\int_{\mathcal{G}(n,i)} \lambda_i(K|S)^p \, dS \right)^{1/p},$$

when $1 \leq i \leq n-1$, and $\Phi_{n,p}(K) = \kappa_n$. Thus $\Phi_{n-i,-n}(K) = \Phi_{n-i}(K)$ is the usual affine quermassintegral of K. (The case when $p = 0$ is also considered in [187, Section 7.2], where it is shown to play a role in an averaged Loomis–Whitney inequality.) They conjecture that for $0 \leq i \leq j \leq n-1$ and $0 > p \geq -n$, we have

$$\kappa_n^j \Phi_{i,p}(K)^{n-j} \leq \kappa_n^i \Phi_{j,p}(K)^{n-i}, \tag{40}$$

with equality for $p > -n$ if and only if K is a ball. The case $p = -n$ is Problem 9.5, where equality is conjectured to hold if and only if K is an ellipsoid. In [187, Theorem 1.5], (40) is proved when $0 > p \geq i - n$, the other cases remaining open.

It is interesting that with one exception, the situation is exactly mirrored in the dual case. Gardner [89, (41), p. 379] introduced what might now be called *dual L_p-moment affine quermassintegrals*, defined for bounded Borel sets C, $0 \leq i \leq n-1$, and $0 < p \leq n$, by

$$\widetilde{\Phi}_{n-i,p}(C) = \frac{\kappa_n}{\kappa_i} \left(\int_{\mathcal{G}(n,i)} \lambda_i(C \cap S)^p \, dS \right)^{1/p},$$

when $1 \leq i \leq n-1$, and $\widetilde{\Phi}_{n,p}(K) = \kappa_n$, and (without stating it as a conjecture) considered the possibility that for $0 \leq i \leq j \leq n-1$ and $0 < p \leq n$, we have

$$\kappa_n^j \widetilde{\Phi}_{i,p}(C)^{n-j} \geq \kappa_n^i \widetilde{\Phi}_{j,p}(C)^{n-i}. \tag{41}$$

In [89, Theorem 7.4], (41) is proved, with precise equality conditions, when $0 < p \leq n-i$, which are just the values corresponding to those in [187, Theorem 1.5]. The exception mentioned above is that in [89, Theorem 7.7], (41) is actually proved to be false when $1 \leq i \leq n-2$, $j = n-1$, $p = n$, and C is a centered convex body that is not an ellipsoid. However, for sections, unlike projections, the case $j = n-1$ is different in an essential way; it corresponds to the case $i = 1$ of [89, Theorem 7.4], where the equality condition indicates this difference. It remains quite possible, then, that (41) is true when $j \leq n-2$.

Problem 9.6

Dafnis and Paouris [67] (see the report for Problem 9.3) also asked if there is a universal constant $c_2 > 0$ such that

$$\Phi_{[i]}(K) \le c_2\sqrt{n/i},$$

for $1 \le i \le n-1$, and showed that this inequality holds with an extra factor of $\log n$ on the right-hand side. It has been verified for some classes of random polytopes by Chasapis and Skarmogiannis [59].

References

[1] A. Abbondandolo and G. Benedetti. On the local systolic optimality of Zoll contact forms. *Geom. Funct. Anal.* **33** (2023), 299–363.

[2] A. Abbondandolo, B. Bramham, U. L. Hryniewicz, and P. A. Salomão. Sharp systolic inequalities for Reeb flows on the three-sphere. *Invent. Math.* **211** (2018), 687–778.

[3] A. Akopyan, A. Balitskiy, R. Karasev, and A. Sharipova. Elementary approach to closed billiard trajectories in asymmetric normed spaces. *Proc. Amer. Math. Soc.* **144** (2016), 4501–13.

[4] A. Akopyan, R. Karasev, and F. Petrov. Bang's problem and symplectic invariants. *J. Symplectic Geom.* **17** (2019), 1579–611.

[5] A. D. Aleksandrov. Mean values of the support function [in Russian]. *Dokl. Akad. Nauk SSSR* **172** (1967), 755–8. English translation: *Soviet Math. Dokl.* **8** (1967), 149–53.

[6] M. Alexander, M. Fradelizi, L. C. García-Lirola, and A. Zvavitch. Geometry and volume product of finite dimensional Lipschitz-free spaces. *J. Funct. Anal.* **280** (2021), 108849, 38 pp.

[7] M. A. Alfonseca, M. Cordier, and D. Ryabogin. On bodies in $\mathbb{R}^5$ with directly congruent projections or sections. *Rev. Mat. Iberoam.* **35** (2019), 1745–62.

[8] D. Alonso-Gutiérrez. On an extension of the Blaschke–Santaló inequality and the hyperplane conjecture. *J. Math. Anal. Appl.* **344** (2008), 292–300.

[9] D. Alonso-Gutiérrez and J. Bastero. *Approaching the Kannan–Lovász–Simonovits and Variance Conjectures*. Lecture Notes in Mathematics 2131, Springer, Cham, 2015.

[10] D. Alonso-Gutiérrez and J. Bastero. Convex inequalities, isoperimetry and spectral gap. In *Advanced Courses of Mathematical Analysis VI*. World Sci. Publ., Hackensack, NJ, 2017, pp. 3–24.

[11] D. Alonso-Gutiérrez and J. Bastero. The variance conjecture on hyperplane projections of the l_p^n balls. *Rev. Mat. Iberoam.* **34** (2018), 879–904.

[12] D. Alonso-Gutiérrez and S. Brazitikos. Reverse Loomis–Whitney inequalities via isotropicity. *Proc. Amer. Math. Soc.* **149** (2021), 817–28.

[13] D. Alonso-Gutiérrez, J. Prochno, and C. Thäle. Large deviations, moderate deviations, and the KLS conjecture. *J. Funct. Anal.* **280** (2021), 108779, 33 pp.

[14] J. C. Álvarez Paiva and F. Balacheff. Contact geometry and isosystolic inequalities. *Geom. Funct. Anal.* **24** (2014), 648–69.

[15] J. C. Álvarez Paiva, F. Balacheff, and K. Tzanev. Isosystolic inequalities for optical hypersurfaces. *Adv. Math.* **301** (2016), 934–72.

[16] S. Artstein-Avidan, A. Giannopoulos, and V. D. Milman. *Asymptotic Geometric Analysis, Part I*. Mathematical Surveys and Monographs 202, American Mathematical Society, Providence, RI, 2015.

[17] S. Artstein-Avidan, R. Karasev, and Y. Ostrover. From symplectic measurements to the Mahler conjecture. *Duke Math. J.* **163** (2014), 2003–22.

[18] S. Artstein-Avidan, B. Klartag, and V. Milman. The Santaló point of a function, and a functional form of the Santaló inequality. *Mathematika* **51** (2004), 33–48.

[19] S. Artstein-Avidan, V. Milman, and Y. Ostrover. The M-ellipsoid, symplectic capacities and volume. *Comment. Math. Helv.* **83** (2008), 359–69.
[20] S. Artstein-Avidan, S. Sadovsky, and R. Sanyal. Geometric inequalities for anti-blocking bodies. *Commun. Contemp. Math.* **25** (2023), Paper No. 2150113, 30 pp.
[21] F. Balacheff, G. Solanes, and K. Tzanev. Projective transformations of convex bodies and volume product. *Int. Math. Res. Not. IMRN* 2024, no. 11, 9054–65.
[22] A. Balitskiy. Equality cases in Viterbo's conjecture and isoperimetric billiard inequalities. *Int. Math. Res. Not. IMRN* 2020, no. 7, 1957–58.
[23] K. Ball. Isometric problems in l_p and sections of convex sets. PhD dissertation, University of Cambridge, Cambridge, 1986.
[24] K. Ball. A remark on the slicing problem. In *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 1745. Springer, Berlin, 2000, pp. 21–26.
[25] K. Ball and V. H. Nguyen. Entropy jumps for isotropic log-concave random vectors and spectral gap. *Studia Math.* **213** (2012), 81–96.
[26] S. Banach. *Theory of Linear Operations*. North-Holland Publishing Co., Amsterdam, 1987. French original: *Théorie des Opérations Linéaires*. Monograf. Mat., Warsaw, 1932.
[27] C. Bandle and M. Marcus. Radial averaging transformations with various metrics. *Pacific J. Math.* **46** (1973) 337–48.
[28] I. Bárány and C. Buchta. Random polytopes in a convex polytope, independence of shape, and concentration of vertices. *Math. Ann.* **297** (1993), 467–97.
[29] F. Barthe and M. Fradelizi. The volume product of convex bodies with many hyperplane symmetries. *Amer. J. Math.* **135** (2013), 311–47.
[30] F. Barthe and P. Wolff. Volume properties of high-dimensional Orlicz balls. In *High Dimensional Probability IX—the Ethereal Volume*. Progr. Probab. 80, Birkhäuser/Springer, Cham, 2023, pp. 75–95.
[31] M. Berezovik and R. Karasev. Symplectic polarity and Mahler's conjecture. Preprint, arXiv:2211.14630v1.
[32] B. Berndtsson. Bergman kernels for Paley–Wiener spaces and Nazarov's proof of the Bourgain–Milman theorem. *Pure Appl. Math. Q.* **18** (2022), 395–409.
[33] B. Berndtsson. Complex integrals and Kuperberg's proof of the Bourgain–Milman theorem. *Adv. Math.* **388** (2021), Paper No. 107927, 10 pp.
[34] B. Berndtsson, V. Mastrantonis, and Y. A. Rubinstein. L^p-polarity, Mahler volumes, and the isotropic constant. *Anal. PDE* **17** (2024), 2179–245.
[35] G. Bianchi and M. Kelly. A Fourier analytic proof of the Blaschke–Santaló inequality. *Proc. Amer. Math. Soc.* **143** (2015), 4901–12.
[36] P. Bizeul. The slicing conjecture via small ball estimates. Preprint, arXiv:2501.06854v1.
[37] Z. Błocki. Cauchy–Riemann meet Monge–Ampère. *Bull. Math. Sci.* **4** (2014), 433–80.
[38] Z. Błocki. A lower bound for the Bergman kernel and the Bourgain–Milman inequality. In *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 2116. Springer, Cham, 2014, pp. 53–63.
[39] Z. Błocki. On Nazarov's complex analytic approach to the Mahler conjecture and the Bourgain–Milman inequality. In *Complex Analysis and Geometry*, Springer Proc. Math. Stat. 144. Springer, Tokyo, 2015, pp. 89–98.
[40] S. Bobkov, B. Klartag, and A. Koldobsky. Estimates for moments of general measures on convex bodies. *Proc. Amer. Math. Soc.* **146** (2018), 4879–88.
[41] S. G. Bobkov and A. Koldobsky. On the central limit property of convex bodies. In *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 1807. Springer, Berlin, 2003, pp. 44–52.
[42] S. G. Bobkov and M. Madiman. Entropy and the hyperplane conjecture in convex geometry. In *Information Theory Proceedings (ISIT), 2010*. IEEE International Symposium, Austin, Texas, 2010, pp. 1438–42.

[43] S. G. Bobkov and M. Madiman. The entropy per coordinate of a random vector is highly constrained under convexity conditions. *IEEE Trans. Inform. Theory* **57** (2011), 4940–54.
[44] S. G. Bobkov and M. Madiman. Reverse Brunn–Minkowski and reverse entropy power inequalities for convex measures. *J. Funct. Anal.* **262** (2012), 3309–39.
[45] S. G. Bobkov and M. Madiman. On the problem of reversibility of the entropy power inequality. In *Limit Theorems in Probability, Statistics and Number Theory*, Springer Proc. Math. Stat. 42. Springer, Heidelberg, 2013, pp. 61–74.
[46] G. Bor, L. Hernández-Lamoneda, V. Jiménez-Desantiago, and L. Montejano. On the isometric conjecture of Banach. *Geom. Topol.* **25** (2021), 2621–42.
[47] J. Bourgain. On high-dimensional maximal functions associated to convex bodies. *Amer. J. Math.* **108** (1986), 1467–76.
[48] J. Bourgain, B. Klartag, and V. Milman. Symmetrization and isotropic constants of convex bodies. In *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 1850. Springer, Berlin, 2004, pp. 101–15.
[49] S. Brazitikos, S. Dann, A. Giannopoulos, and A. Koldobsky. On the average volume of sections of convex bodies. *Israel J. Math.* **222** (2017), 921–47.
[50] S. Brazitikos, A. Giannopoulos, P. Valettas, and B. Vritsiou. *Geometry of Isotropic Convex Bodies*. Mathematical Surveys and Monographs 196, American Mathematical Society, Providence, RI, 2014.
[51] S. Brazitikos and D. Liakopoulos. On a version of the slicing problem for the surface area of convex bodies. *Trans. Amer. Math. Soc.* **375** (2022), 5561–86.
[52] D. Bucur and I. Fragalà. Reverse Faber–Krahn and Mahler inequalities for the Cheeger constant. *Proc. Roy. Soc. Edinburgh Sect. A* **148** (2018), 913–37.
[53] P. Buser. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)* **15** (1982), 213–30.
[54] S. Campi, A. Colesanti, and P. Gronchi. A note on Sylvester's problem for random polytopes in a convex body. *Rend. Istit. Mat. Univ. Trieste* **31** (1999), 79–94.
[55] M. Campos, P. van Hintum, R. Morris, and M. Tiba. Towards Hadwiger's conjecture via Bourgain slicing. *Int. Math. Res. Not. IMRN* 2024, no. 10, 8282–95.
[56] G. D. Chakerian and P. Filliman. The measures of the projections of a cube. *Studia Sci. Math. Hungar.* **21** (1986), 103–10.
[57] S. H. Chan and I. Pak. Linear extensions of finite posets. Preprint.
[58] G. Chasapis, A. Giannopoulos, and D. Liakopoulos. Estimates for measures of lower dimensional sections of convex bodies. *Adv. Math.* **306** (2017), 880–904.
[59] G. Chasapis and N. Skarmogiannis. Affine quermassintegrals of random polytopes. *J. Math. Anal. Appl.* **479** (2019), 546–68.
[60] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in Analysis (papers dedicated to Salomon Bochner, 1969)*. Princeton Univ. Press, Princeton, NJ, 1970, pp. 195–99.
[61] Y. Chen. An almost constant lower bound of the isoperimetric coefficient in the KLS conjecture. *Geom. Funct. Anal.* **31** (2021), 34–61.
[62] Y. Chen. CRM Nirenberg Lecture in Geometric Analysis, October 6, 2021. Available via `https://www.youtube.com/watch?v=35NCkJBjKWc`
[63] B. Cousins and S. Vempala. Gaussian cooling and $O^*(n^3)$ algorithms for volume and Gaussian volume. *SIAM J. Comput.* **47** (2018), 1237–73.
[64] D. Cristofero–Gardiner and R. Hind. On the agreement of symplectic capacities in high dimension. Preprint, arXiv:2307.12125v1.
[65] B. Dadoun, M. Fradelizi, O. Guédon, and P.-A. Zitt. Asymptotics of the inertia moments and the variance conjecture in Schatten balls. *J. Funct. Anal.* **284** (2023), Paper No. 109741, 32 pp.
[66] N. Dafnis and G. Paouris. Small ball probability estimates, ψ_2-behavior and the hyperplane conjecture. *J. Funct. Anal.* **258** (2010), 1933–64.

[67] N. Dafnis and G. Paouris. Estimates for the affine and dual affine quermassintegrals of convex bodies. *Illinois J. Math.* **56** (2012), 1005–21.
[68] L. Dalla and D. G. Larman. Volumes of a random polytope in a convex set. In *Applied Geometry and Discrete Mathematics*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 4. Amer. Math. Soc., Providence, RI, 1991, pp. 175–80.
[69] N. De Ponti and A. Mondino. Sharp Cheeger–Buser type inequalities in RCD(K,∞) spaces. *J. Geom. Anal.* **31** (2021), 2416–38.
[70] Y. Disser and S. S. Skiena. Geometric reconstruction problems. Chapter 34 in *Handbook of discrete and computational geometry*, third edition, ed. by J. E. Goodman, J. O'Rourke, and C. D.Tóth, CRC Press, Boca Raton, FL, 2018.
[71] P. Dulio, M. Longinetti, C. Peri, and A. Venturi. Sharp affine stability estimates for Hammer's problem. *Adv. in Appl. Math.* **41** (2008), 27–51.
[72] O. Edtmair. Disk-like surfaces of section and symplectic capacities. *Geom. Funct. Anal.* **34** (2024), 1399–459.
[73] R. Eldan. Thin shell implies spectral gap up to polylog via a stochastic localization scheme. *Geom. Funct. Anal.* **23** (2013), 532–69.
[74] R. Eldan. Analysis of high-dimensional distributions using pathwise methods. In *ICM—International Congress of Mathematicians*, Vol. 6, EMS Press, Berlin, 2023, Sections 12–14, pp. 4246–70.
[75] R. Eldan and B. Klartag. Approximately Gaussian marginals and the hyperplane conjecture. In *Concentration, Functional Inequalities and Isoperimetry*, Contemp. Math. 545. Amer. Math. Soc., Providence, RI, 2011, pp. 55–68.
[76] D. Faifman, C. Vernicos, and C. Walsh. Volume growth of Funk geometry and the flags of polytopes. Preprint, arXiv:2306.09268v1.
[77] H. Federer. *Geometric Measure Theory*. Springer, New York, 1969.
[78] A. Fish, F. Nazarov, D. Ryabogin, and A. Zvavitch. The unit ball is an attractor of the intersection body operator. *Adv. Math.* **226** (2011), 2629–42.
[79] M. Fradelizi, N. Gozlan, S. Sadovsky, and S. Zugmeyer. Transport-entropy and functional forms of Blaschke–Santaló inequalities. *Rev. Mat. Iberoam.* **40** (2024), 1917–52.
[80] M. Fradelizi, A. Hubard, M. Meyer, E. Roldán-Pensado, and A. Zvavitch. Equipartitions and Mahler volumes of symmetric convex bodies. *Amer. J. Math.* **144** (2022), 1201–19.
[81] M. Fradelizi and F. Marín Sola. Entropy, slicing problem and functional Mahler's conjecture. Preprint, arXiv:2406.07406v1.
[82] M. Fradelizi and M. Meyer. Some functional inverse Santaló inequalities. *Adv. Math.* **218** (2008), 1430–52.
[83] M. Fradelizi, M. Meyer, and A. Zvavitch. Volume product. In *Harmonic Analysis and Convexity*, Adv. Anal. Geom. 9. De Gruyter, Berlin, 2023, pp. 163–222.
[84] M. Fradelizi and E. Nakhle. The functional form of Mahler's conjecture for even log-concave functions in dimension 2. *Int. Math. Res. Not. IMRN* 2023, no. 12, 10067–97.
[85] M. Fradelizi and E. Nakhle. On Mahler's conjecture for even s-concave functions in dimensions 1 and 2. Preprint, arXiv:2412.12372v1.
[86] D. Fresen. The floating body and the hyperplane conjecture. *Arch. Math. (Basel)* **98** (2012), 389–97.
[87] D. Fresen. A simplified proof of CLT for convex bodies. *Proc. Amer. Math. Soc.* **149** (2021), 345–54.
[88] A. Freyer and M. Henk. Polynomial bounds in Koldobsky's discrete slicing problem. *Proc. Amer. Math. Soc.* **152** (2024), 3063–74.
[89] R. J. Gardner. The dual Brunn–Minkowski theory for bounded Borel sets: dual affine quermassintegrals and inequalities. *Adv. Math.* **216** (2007), 358–86.
[90] R. J. Gardner, D. Ryabogin, V. Yaskin, and A. Zvavitch. A problem of Klee on inner section functions of convex bodies. *J. Differential Geom.* **91** (2012), 261–79.

[91] A. Giannopoulos. On the mean value of the area of a random polygon in a plane convex body. *Mathematika* **39** (1992), 279–90.
[92] A. Giannopoulos. Geometry of isotropic convex bodies and the slicing problem. Lecture notes for the introductory workshop of the program *Geometric Functional Analysis and Applications*, MSRI, 2017. Available via `http://users.uoa.gr/~apgiannop/`.
[93] A. Giannopoulos and A. Koldobsky. Variants of the Busemann–Petty problem and of the Shephard problem. *Int. Math. Res. Not. IMRN* 2017, no. 3, 921–43.
[94] A. Giannopoulos and A. Koldobsky. Volume difference inequalities. *Trans. Amer. Math. Soc.* **370** (2018), 4351–72.
[95] A. Giannopoulos, A. Koldobsky, and A. Zvavitch. Inequalities for the Radon transform on convex sets. *Int. Math. Res. Not. IMRN* 2022, no. 18, 13984–14007.
[96] A. Giannopoulos, A. Koldobsky, and A. Zvavitch. Inequalities for sections and projections of convex bodies. In *Harmonic Analysis and Convexity*, Adv. Anal. Geom. 9. De Gruyter, Berlin, 2023, pp. 223–56.
[97] A. Giannopoulos, G. Paouris, and B. Vritsiou. The isotropic position and the reverse Santaló inequality. *Israel J. Math.* **203** (2014), 1–22.
[98] P. Gritzmann, B. Langfeld, and M. Wiegelmann. Uniqueness in discrete tomography: three remarks and a corollary. *SIAM J. Discrete Math.* **25** (2011), 1589–99.
[99] M. Gromov and V. D. Milman. A topological application of the isoperimetric inequality. *Amer. J. Math.* **105** (1983), 843–54.
[100] Q.-Y. Guan. A note on Bourgain's slicing problem. Preprint, arXiv:2412.09075v1.
[101] O. Guédon and E. Milman. Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures. *Geom. Funct. Anal.* **21** (2011), 1043–68.
[102] H. W. Guggenheimer. *The Analytic Geometry of the Unsymmetric Minkowski Plane*. University of Minnesota, Institute of Technology, School of Mathematics, Minneapolis, 1967.
[103] J. Gutt, M. Hutchings, and V. G. B. Ramos. Examples around the strong Viterbo conjecture. *J. Fixed Point Theory Appl.* **24** (2022), Paper No. 41, 22 pp.
[104] J. Haddad and A. Koldobsky. Radon transforms with small derivatives and distance inequalities for convex bodies. Preprint, arXiv:2312.16923v1.
[105] P. Haim-Kislev. On the symplectic size of convex polytopes. *Geom. Funct. Anal.* **29** (2019), 440–63.
[106] P. Haim-Kislev and Y. Ostrover. A counterexample to Viterbo's conjecture. Preprint, arXiv:2405.16513v1.
[107] O. Hanner. Intersections of translates of convex bodies. *Math. Scand.* **4** (1956), 65–87.
[108] A. B. Hansen and A. Lima. The structure of finite-dimensional Banach spaces with the 3.2. intersection property. *Acta Math.* **146** (1981), 1–23.
[109] E. M. Harrell, A. Henrot, and J. Lamboley. On the local minimizers of the Mahler volume. *J. Convex Anal.* **22** (2015), 809–25.
[110] D. Hermann. Non-equivalence of symplectic capacities for open sets with restricted contact type boundary. *Prépublication d'Orsay* 32 (29/4/1998), available at `http://www.math.u-psud.fr/~biblio/pub/1998/fic/ppo_1998_32.ps`.
[111] R. Howard, F. Nazarov, D. Ryabogin, and A. Zvavitch. Determining starlike bodies by the perimeters of their central sections. Preprint.
[112] H. Huang, B. A. Slomka, T. Tkocz, and B. Vritsiou. Improved bounds for Hadwiger's covering problem via thin-shell estimates. *J. Eur. Math. Soc. (JEMS)* **24** (2022), 1431–48.
[113] D. Hug. Nakajima's problem for general convex bodies. *Proc. Amer. Math. Soc.* **137** (2009), 255–63.
[114] H. Iriyeh and M. Shibata. Symmetric Mahler's conjecture for the volume product in the 3-dimensional case. *Duke Math. J.* **169** (2020), 1077–134.

[115] H. Iriyeh and M. Shibata. Minimal volume product of three dimensional convex bodies with various discrete symmetries. *Discrete Comput. Geom.* **68** (2022), 738–73.
[116] H. Iriyeh and M. Shibata. Minimal volume product of convex bodies with certain discrete symmetries and its applications. *Int. Math. Res. Not. IMRN* 2023, no. 21, 18001–34.
[117] H. Iriyeh and M. Shibata. Minimal volume product of three-dimensional convex bodies invariant under certain groups of order four. Preprint, arXiv:2409.16785v1.
[118] M. N. Ivaki. The second mixed projection problem and the projection centroid conjectures. *J. Funct. Anal.* **272** (2017), 5144–61.
[119] M. N. Ivaki. A local uniqueness theorem for minimizers of Petty's conjectured projection inequality. *Mathematika* **64** (2018), 1–19.
[120] S. V. Ivanov. On two-dimensional minimal fillings. *St. Petersburg Math. J.* **13** (2002), 17–25.
[121] A. Jambulapati, Y. T. Lee, and S. S. Vempala. A slightly improved bound for the KLS constant. Preprint, arXiv:2208.11644v2.
[122] D. Jerison. The two hyperplane conjecture. *Acta Math. Sin. (Engl. Ser.)* **35** (2019), 728–48.
[123] H. Jiang, Y. T. Lee, and S. S. Vempala. A generalized central limit conjecture for convex bodies. In *Geometric Aspects of Functional Analysis, Vol. II*, Lecture Notes in Mathematics 2266. Springer, Cham, 2020, pp. 1–41.
[124] G. Kalai. The number of faces of centrally-symmetric polytopes. *Graphs Combin.* **5** (1989), 389–91.
[125] R. Kannan, L. Lovász, and M. Simonovits. Isoperimetric problems for convex bodies and a localization lemma. *Discrete Comput. Geom.* **13** (1995), 541–59.
[126] R. Karasev. Mahler's conjecture for some hyperplane sections. *Israel J. Math.* **241** (2021), 795–815.
[127] R. Karasev and A. Sharipova. Convex bodies with all characteristics planar. *Int. Math. Res. Not. IMRN* 2024, no. 10, 8104–21.
[128] L. Kathuria and M. Raka. On conjectures of Minkowski and Woods for $n = 10$. *Proc. Indian Acad. Sci. Math. Sci.* **132** (2022), Paper No. 45, 27 pp.
[129] J. Kim. Minimal volume product near Hanner polytopes. *J. Funct. Anal.* **266** (2014), 2360–402.
[130] J. Kim and S. Reisner. Local minimality of the volume-product at the simplex. *Mathematika* **57** (2011), 121–34.
[131] B. Klartag. An isomorphic version of the slicing problem. *J. Funct. Anal.* **218** (2005), 372–94.
[132] B. Klartag. On convex perturbations with a bounded isotropic constant. *Geom. Funct. Anal.* **16** (2006), 1274–90.
[133] B. Klartag. A central limit theorem for convex sets. *Invent. Math.* **168** (2007), 91–131.
[134] B. Klartag. A Berry–Esseen type inequality for convex bodies with an unconditional basis. *Probab. Theory Related Fields* **145** (2009), 1–33.
[135] B. Klartag. Convex geometry and waist inequalities. *Geom. Funct. Anal.* **27** (2017), 130–64.
[136] B. Klartag. Isotropic constants and Mahler volumes. *Adv. Math.* **330** (2018), 74–108.
[137] B. Klartag. On Yuansi Chen's work on the KLS conjecture. Lecture notes from a winter school at the Hausdorff Institute, January 2021. Available at `https://www.weizmann.ac.il/math/klartag/sites/math.klartag/files/uploads/chen`
[138] B. Klartag. Logarithmic bounds for isoperimetry and slices of convex sets. *Ars Inven. Anal.* 2023, Paper No. 4, 17 pp.
[139] B. Klartag. Notes on Guan's bound. Available at `https://tinyurl.com/3fwp3u22`
[140] B. Klartag and A. Koldobsky. An example related to the slicing inequality for general measures. *J. Funct. Anal.* **274** (2018), 2089–112.
[141] B. Klartag and G. Kozma. On the hyperplane conjecture for random convex sets. *Israel J. Math.* **170** (2009), 253–68.
[142] B. Klartag and J. Lehec. Bourgain's slicing problem and KLS isoperimetry up to polylog. *Geom. Funct. Anal.* **32** (2022), 1134–59.

[143] B. Klartag and J. Lehec. Affirmative resolution of Bourgain's slicing problem using Guan's bound. Preprint, arXiv:2412.15044v1.
[144] B. Klartag and J. Lehec. Isoperimetric inequalities in high-dimensional convex sets. *Bull. Amer. Math. Soc.*, to appear. Preprint, arXiv:2406.01324v1.
[145] B. Klartag and G. Livshyts. The lower bound for Koldobsky's slicing inequality via random rounding. In *Geometric Aspects of Functional Analysis, Vol. II*, Lecture Notes in Mathematics 2266. Springer, Cham, 2020, pp. 43–63.
[146] B. Klartag and E. Milman. Centroid bodies and the logarithmic Laplace transform—a unified approach. *J. Funct. Anal.* **262** (2012), 10–34.
[147] B. Klartag and V. Milman. The slicing problem by Bourgain. In *Analysis at Large: Dedicated to the Life and Work of Jean Bourgain*. Springer, Cham, Switzerland, 2022, pp. 203–31.
[148] B. Klartag and E. Werner. Some open problems in asymptotic geometric analysis. *Notices Amer. Math. Soc.* **65** (2018), 657–59.
[149] A. Koldobsky. Stability in the Busemann–Petty and Shephard problems. *Adv. Math.* **228** (2011), 2145–61.
[150] A. Koldobsky. Estimates for measures of sections of convex bodies. In *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 2116. Springer, Cham, 2014, pp. 261–71.
[151] A. Koldobsky. Slicing inequalities for measures of convex bodies. *Adv. Math.* **283** (2015), 473–88.
[152] A. Koldobsky. Slicing inequalities for subspaces of L_p. *Proc. Amer. Math. Soc.* **144** (2016), 787–95.
[153] A. Koldobsky. Measures of sections of convex bodies. In *Convexity and Concentration*, IMA Vol. Math. Appl. 161. Springer, New York, 2017, pp. 565–75.
[154] A. Koldobsky, G. Paouris, and A. Zvavitch. Measure comparison and distance inequalities for convex bodies. *Indiana Univ. Math. J.* **71** (2022), 391–407.
[155] A. Koldobsky, M. Roysdon, and A. Zvavitch. Comparison problems for Radon transforms. *Adv. Math.* **434** (2023), Paper No. 109336, 32 pp.
[156] A. Koldobsky, C. Saroglou, and A. Zvavitch. Estimating volume and surface area of a convex body via its projections or sections. *Studia Math.* **244** (2019), 245–64.
[157] A. Koldobsky and A. Zvavitch. An isomorphic version of the Busemann–Petty problem for arbitrary measures. *Geom. Dedicata* **174** (2015), 261–77.
[158] A. V. Kolesnikov and E. Milman. Remarks on the KLS conjecture and Hardy-type inequalities. In *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 2116, Springer, Cham, 2014, pp. 273–92.
[159] A. V. Kolesnikov and E. Milman. The KLS isoperimetric conjecture for generalized Orlicz balls. *Ann. Probab.* **46** (2018), 3578–615.
[160] H. König and A. Koldobsky. On the maximal perimeter of sections of the cube. *Adv. Math.* **346** (2019), 773–804.
[161] G. Kuperberg. From the Mahler conjecture to Gauss linking integrals. *Geom. Funct. Anal.* **18** (2008), 870–92.
[162] R. Latała and J. O. Wojtaszczyk. On the infimum convolution inequality. *Studia Math.* **189** (2008), 147–87.
[163] M. Ledoux. Spectral gap, logarithmic Sobolev constant, and geometric bounds. In *Surveys in Differential Geometry. Vol. IX*, Surv. Differ. Geom. 9. Int. Press, Somerville, MA, 2004, pp. 219–40.
[164] Y. T. Lee and S. S. Vempala. Eldan's stochastic localization and the KLS hyperplane conjecture: an improved lower bound for expansion. In *58th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2017*. IEEE Computer Soc., Los Alamitos, CA, 2017, pp. 998–1007.
[165] Y. T. Lee and S. S. Vempala. The Kannan–Lovász–Simonovits conjecture. In *Current Developments in Mathematics 2017*. Int. Press, Somerville, MA, 2019, pp. 1–36.

[166] Y. T. Lee and S. S. Vempala. Eldan's stochastic localization and the KLS conjecture: isoperimetry, concentration and mixing. *Ann. of Math.* (2) **199** (2024), 1043–92.
[167] G. P. Leonardi. An overview on the Cheeger problem. In *New Trends in Shape Optimization*, Internat. Ser. Numer. Math. 166. Birkhäuser/Springer, Cham, 2015, pp. 117–39.
[168] D. Liakopoulos. Inequalities for the quermassintegrals of sections of convex bodies. Preprint, arXiv:2302.10568.
[169] L. Lovász and M. Simonovits. Random walks in a convex body and an improved volume algorithm. *Random Structures Algorithms* **4** (1993), 359–412.
[170] B. Mackey. $SO(2)$-congruent projections of convex bodies with rotation about the origin. *Proc. Amer. Math. Soc.* **143** (2015), 1739–44.
[171] M. Madiman, P. Nayar, and T. Tkocz. Sharp moment-entropy inequalities and capacity bounds for log-concave distributions. *IEEE Trans. Inform. Theory* **67** (2021), 81–94.
[172] A. Magazinov. A proof of a conjecture by Haviv, Lyubashevsky and Regev on the second moment of a lattice Voronoi cell. *Adv. Geom.* **20** (2020), 117–20.
[173] K. Mahler. Ein Übertragungsprinzip für konvexe Körper. *Časopis Pěst. Mat. Fys.* **68** (1939), 93–102.
[174] A. Marsiglietti and V. Kostina. New connections between the entropy power inequality and geometric inequalities. In *Proc. 2018 IEEE International Symposium on Information Theory, Vail, CO, 2018*, pp. 1978–82.
[175] J. Martín-Goñi. A note on the reduction of the slicing problem to centrally symmetric bodies. *J. Math. Anal. App.* **525** (2023), Paper No. 127135, 15 pp.
[176] V. Mastrantonis and Y. A. Rubinstein. The Nazarov proof of the non-symmetric Bourgain-Milman inequality. *Indiana Univ. Math. J.* **73** (2024), 911–53.
[177] V. G. Maz'ya. Classes of domains and imbedding theorems for function spaces. *Soviet Math. Dokl.* **1** (1960), 882–85.
[178] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*, third edition. Oxford University Press, Oxford, 2017.
[179] C. T. McMullen. Minkowski's conjecture, well-rounded lattices and topological dimension. *J. Amer. Math. Soc.* **18** (2005), 711–34.
[180] M. W. Meckes. Volumes of symmetric random polytopes. *Arch. Math. (Basel)* **82** (2004), 85–96.
[181] M. W. Meckes. Magnitude and Holmes–Thompson intrinsic volumes of convex bodies. *Canad. Math. Bull.* **66** (2023), 854–67.
[182] M. Meyer. Une caractérisation volumique de certains espaces normés de dimension finie. *Israel J. Math.* **55** (1986), 317–26.
[183] M. Meyer and S. Reisner. The isotropy constant and boundary properties of convex bodies. *Proc. Amer. Math. Soc.* **144** (2016), 3935–47.
[184] E. Milman. On the role of convexity in isoperimetry, spectral gap and concentration. *Invent. Math.* **177** (2009), 1–43.
[185] E. Milman. Isoperimetric bounds on convex manifolds. *Contemp. Math.* **545** (2011), 195–208.
[186] E. Milman, S. Shabelman, and A. Yehudayoff. Fixed and periodic points of the intersection body operator. Preprint, arXiv:2408.08171v3.
[187] E. Milman and A. Yehudayoff. Sharp isoperimetric inequalities for affine quermassintegrals. *J. Amer. Math. Soc.* **36** (2023), 1061–101.
[188] L. Montejano. Convex bodies with affinely equivalent projections and affine bodies of revolution. *J. Convex Anal.* **28** (2021), 871–77.
[189] L. Montejano. Convex bodies all whose sections (projections) are equal. In *European Congress of Mathematics*. EMS Press, Berlin, 2023, pp. 857–83.
[190] S. Myroshnychenko and D. Ryabogin. On polytopes with congruent projections or sections. *Adv. Math.* **325** (2018), 482–504.

[191] S. Myroshnychenko, D. Ryabogin, and C. Saroglou. Star bodies with completely symmetric sections. *Int. Math. Res. Not. IMRN* 2019, no. 10, 3015–31.
[192] S. Nakamura and H.Tsuji. Hypercontractivity beyond Nelson's time and its applications to Blaschke-Santaló inequality and inverse Santaló inequality. Preprint, arXiv:2212.02866v1.
[193] F. Nazarov. The Hörmander proof of the Bourgain–Milman theorem. In *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 2050. Springer, Heidelberg, 2012, pp. 335–43.
[194] F. Nazarov, F. Petrov, D. Ryabogin, and A. Zvavitch. A remark on the Mahler conjecture: local minimality of the unit cube. *Duke Math. J.* **154** (2010), 419–30.
[195] F. Nazarov, D. Ryabogin, and A. Zvavitch. Non-uniqueness of convex bodies with prescribed volumes of sections and projections. *Mathematika* **59** (2013), 213–21.
[196] F. Nazarov, D. Ryabogin, and A. Zvavitch. An asymmetric convex body with maximal sections of constant volume. *J. Amer. Math. Soc.* **27** (2014), 43–68.
[197] Online Asymptotic Geometric Analysis Seminar. Organized by G. Livshyts, L. Rotem, D. Ryabogin, and K. Tikhorimov. Talks available via `https://glivshyts6.math.gatech.edu/AGAonline.html`.
[198] O. Ortega-Moreno and F. E. Schuster. Fixed points of Minkowski valuations. *Adv. Math.* **392** (2021), Paper No. 108017.
[199] Y. Ostrover. When symplectic topology meets Banach space geometry. In *Proceedings of the International Congress of Mathematicians—Seoul 2014, Vol. II*. Kyung Moon Sa, Seoul, 2014, pp. 959–81.
[200] G. Paouris. On the isotropic constant of non-symmetric convex bodies. In *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics 1745. Springer, Berlin, 2000, pp. 239–43.
[201] G. Paouris and P. Pivovarov. Small-ball probabilities for the volume of random convex sets. *Discrete Comput. Geom.* **49** (2013), 601–46.
[202] A. Papadopoulos and M. Troyanov. Weak Finsler structures and the Funk weak metric. *Math. Proc. Cambridge Philos. Soc.* **147** (2009), 419–37.
[203] L. E. Payne and H. F. Weinberger. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.* **5** (1960), 286–92.
[204] R. E. Pfiefer. The historical development of J. J. Sylvester's four point problem. *Math. Mag.* **62** (1989), 309–17.
[205] J. Prochno, C. Thäle, and N. Turchi. The isotropic constant of random polytopes with vertices on convex surfaces. *J. Complexity* **54** (2019), 101394, 17 pp.
[206] P. M. Pu. Some inequalities in certain nonorientable Riemannian manifolds. *Pacific J. Math.* **2** (1952), 55–71.
[207] L. A. Rademacher. A simplicial polytope that maximizes the isotropic constant must be a simplex. *Mathematika* **62** (2016), 307–20.
[208] J. Radke and B. Vritsiou. On the thin-shell conjecture for the Schatten classes. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020), 87–119.
[209] J. Rebollo Bueno. Stochastic reverse isoperimetric inequalities in the plane. *Geom. Dedicata* **215** (2021), 401–13.
[210] O. Regev and N. Stephens-Davidowitz. A reverse Minkowski theorem. In *STOC'17—Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*. ACM, New York, 2017, pp. 941–53.
[211] S. Reisner, C. Schütt, and E. Werner. Mahler's conjecture and curvature. *Int. Math. Res. Not. IMRN* 2012, no. 1, 1–16.
[212] C. A. Rogers. A single integral inequality. *J. London Math. Soc.* **32** (1957), 102–8.
[213] A. Ros. The isoperimetric problem. In *Global Theory of Minimal Surfaces*, Clay Math. Proc. 2. Amer. Math. Soc., Providence, RI, 2005, pp. 175–209.
[214] O. S. Rothaus. Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators. *J. Functional Analysis* **42** (1981), 110–20.

[215] B. Rubin. The lower dimensional Busemann–Petty problem for bodies with the generalized axial symmetry. *Israel J. Math.* **173** (2009), 213–33.
[216] D. Rudolf. Viterbo's conjecture as a worm problem. *Monatsh. Math.* **201** (2023), 217–87.
[217] D. Ryabogin. On the continual Rubik's cube. *Adv. Math.* **231** (2012), 3429–44.
[218] D. Ryabogin. On a functional equation related to convex bodies with $SU(2)$-congruent projections. *Geom. Dedicata* **190** (2017), 151–56.
[219] D. Ryabogin. On symmetries of projections and sections of convex bodies. In *Discrete Geometry and Symmetry*, Springer Proc. Math. Stat. 234. Springer, Cham, 2018, pp. 297–309.
[220] D. Ryabogin and V. Yaskin. On counterexamples in questions of unique determination of convex bodies. *Proc. Amer. Math. Soc.* **141** (2013), 2869–74.
[221] D. Ryabogin and A. Zvavitch. Analytic methods in convex geometry. In *Analytical and Probabilistic Methods in the Geometry of Convex Bodies*, IMPAN Lecture Notes 2. Polish Acad. Sci. Inst. Math., Warsaw, 2014, pp. 87–183.
[222] C. Saroglou. Characterizations of extremals for some functionals on convex bodies. *Canad. J. Math.* **62** (2010), 1404–18.
[223] C. Saroglou. Volumes of projection bodies of some classes of convex bodies. *Mathematika* **57** (2011), 329–53.
[224] C. Saroglou. Shadow systems: remarks and extensions. *Arch. Math. (Basel)* **100** (2013), 389–99.
[225] C. Saroglou. On the shape of a convex body with respect to its second projection body. *Adv. in Appl. Math.* **67** (2015), 55–74.
[226] C. Saroglou and A. Zvavitch. Iterations of the projection body operator and a remark on Petty's conjectured projection inequality. *J. Funct. Anal.* **272** (2017), 613–30.
[227] R. Schneider. *Convex Bodies: the Brunn–Minkowski Theory*, second edition. Cambridge University Press, Cambridge, 2014.
[228] U. Shapira and B. Weiss. Stable lattices and the diagonal group. *J. Eur. Math. Soc. (JEMS)* **18** (2016), 1753–67.
[229] P. Stavrakakis and P. Valettas. On the geometry of log-concave probability measures with bounded log-Sobolev constant. In *Asymptotic Geometric Analysis*, Fields Inst. Commun. 68. Springer, New York, 2013, pp. 359–80.
[230] T. Tao. Open question: the Mahler problem on convex bodies. `https://terrytao.wordpress.com/2007/03/08/open-problem-the-mahler-conjecture-on-convex-bodies/`
[231] T. Tao. *Structure and Randomness. Pages from Year One of a Mathematical Blog.* American Mathematical Society, Providence, RI, 2008.
[232] M. C. H. Tointon. The Mahler conjecture in two dimensions via the probabilistic method. *Amer. Math. Monthly* **125** (2018), 820–28.
[233] C. Vincze and A. Nagy. Generalized conic functions of hv-convex planar sets: continuity properties and relations to X-rays. *Aequationes Math.* **89** (2015), 1015–30.
[234] C. Viterbo. Metric and isoperimetric problems in symplectic geometry. *J. Amer. Math. Soc.* **13** (2000), 411–31.
[235] B. Vritsiou. Selberg-type integrals and the variance conjecture for the operator norm. Preprint, arXiv:1805.03129v1.
[236] D. Wu. The isomorphic Busemann–Petty problem for s-concave measures. *Geom. Dedicata* **204** (2020), 131–48.
[237] V. Yaskin. On perimeters of sections of convex polytopes. *J. Math. Anal. Appl.* **371** (2010), 447–53.
[238] N. Zhang. On bodies with congruent sections or projections. *J. Differential Equations* **265** (2018), 2064–75.

[239] D. Zou and G. Xiong. New affine inequalities and projection mean ellipsoids. *Calc. Var. Partial Differential Equations* **58** (2019), Paper No. 44, 18 pp.

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